Lecture Notes On

Mathematics for Business, Economics and Social Sciences

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Contents

Chapter 1

Functions

Equations: We can perfom the same operation on both sides of an equality:

$$\begin{aligned} 8x - 2 &= 5x + 7 && \text{Add } 2 \text{ to both sides} \\ 8x &= 5x + 9 && \text{Subtract } 5x \text{ from both sides} \\ 3x &= 9 && \text{Multiply both sides by } \tfrac{1}{3} \\ x &= 3 \end{aligned}$$

Example 1–1: Solve the equation $\dfrac{2x}{2x+5} = \dfrac{3}{4}$

Solution:

$$\begin{aligned} \frac{2x}{2x+5} &= \frac{3}{4} \\ 8x &= 6x + 15 \\ 2x &= 15 \\ x &= \frac{15}{2} \end{aligned}$$

Example 1–2: Solve the equation $\left|3x - 12\right| = 27$

Solution: Using the definition of absolute value, we get

$$\begin{aligned} 3x - 12 &= 27 && \text{or} && -3x + 12 &= 27 \\ 3x &= 39 && \text{or} && -3x &= 15 \\ x &= 13 && \text{or} && x &= -5 \end{aligned}$$

Intervals:

- Closed interval: $\left[a,\, b\right] = \{x : a \leqslant x \leqslant b\}$
- Open interval: $\left(a,\, b\right) = \{x : a < x < b\}$
- Half-open interval: $\left(a,\, b\right] = \{x : a < x \leqslant b\}$
- Unbounded interval: $\left(a,\, \infty\right) = \{x : a < x\}$

We will use $\mathbb{R}$ to denote all real numbers, in other words the interval $\left(-\infty,\, \infty\right)$.

Inequalities: Inequalities are similar to equations. We can add the same quantity to both sides, but if we multiply by a negative number, the direction of the inequality is reversed.

Example 1–3: Solve the inequality $7x - 5 \leqslant 30$.

Solution:

$$\begin{aligned} 7x - 5 &\leqslant 23 \\ 7x &\leqslant 28 \\ x &\leqslant 4 \\ x &\in \big(-\infty, 4\big] \end{aligned}$$

Example 1–4: Solve the inequality $\big|x + 10\big| < 11$.

Solution:

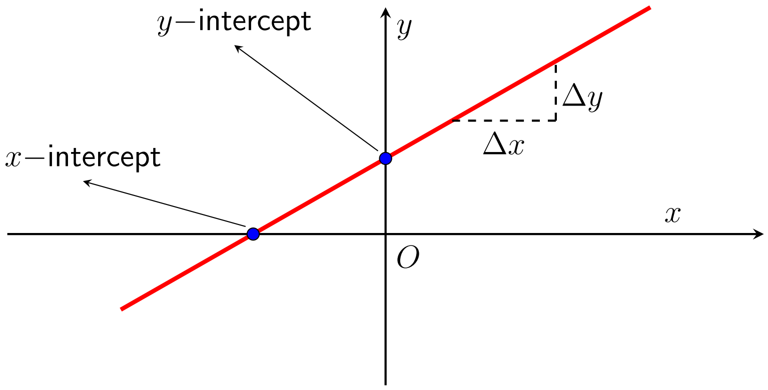
$$\begin{array}{ccccc} -11 & < & x + 10 & < & 11 \\ -21 & < & x & < & 1 \end{array}$$
$$x \in \big(-21, 1\big)$$

Example 1–5: Solve the inequality $\big|x + 10\big| > 11$.

Solution:

$$\begin{array}{rclcrcl} x + 10 & > & 11 & \text{or} & x + 10 & < & -11 \\ x & > & 1 & \text{or} & x & < & -21 \\ x & \in & \big(-\infty, -21\big) & \text{or} & x & \in & \big(1, \infty\big) \end{array}$$

Lines on the Plane:



Slope of a line is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$x = 0$ gives the $y-$intercept and $y = 0$ gives the $x-$intercept.

- Slope-intercept equation: $y = mx + n$.
- Point-slope equation: $y - y_1 = m(x - x_1)$.

If we are given two points on a line or one point and the slope, we can find the equation of the line.

- If two lines are parallel: $m_1 = m_2$.
- If two lines are perpendicular: $m_1 \cdot m_2 = -1$.

The equation $x = c$ gives a vertical line and $y = c$ gives a horizontal line.

Example 1–6: Find the equation of the line passing through the points $(2,9)$ and $(4,13)$.

Solution: Let's find the slope first: $m = \dfrac{13-9}{4-2} = 2$

Now, let's use the point-slope form of a line equation using the point $(2,9)$:

$$(y-9) = 2(x-2)$$

$$y = 2x+5$$

If we use $(4,13)$, we will obtain the same result:

$$(y-13) = 2(x-4)$$

$$y = 2x+5$$

Example 1–7: Find the equation of the line passing through the point $(2,4)$ and parallel to the line $3x+5y=1$.

Solution: If we rewrite the line equation as: $y = -\dfrac{3}{5}x + \dfrac{1}{5}$

we see that $m = -\dfrac{3}{5}$.

Therefore: $y-4 = -\dfrac{3}{5}(x-2)$

$$y = -\frac{3}{5}x + \frac{26}{5}$$

or $3x+5y=26$.

Example 1–8: Find the equation of the line passing through the points $(24,0)$ and $(8,-6)$.

Solution: The slope is:

$$m = \frac{-6-0}{8-24} = \frac{-6}{-16} = \frac{3}{8}$$

Using point-slope equation, we find:

$$y-0 = \frac{3}{8}(x-24) \quad \Rightarrow \quad y = \frac{3}{8}x - 9$$

In other words: $3x-8y=72$.

Example 1–9: Find the equation of the line passing through origin and parallel to the line $2y-8x-12=0$.

Solution: If we rewrite the line equation as:

$$y = 4x+6,$$

we see that $m=4$. Therefore:

$$y-0 = 4(x-0)$$

$$y = 4x.$$

Note that a line through origin has zero intercept.

Function: A function f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number, denoted by $f(x)$.
For example:

$$\begin{aligned} f(x) &= x^2 \\ f(x) &= 7x+2 \\ f(x) &= |x| \\ f(x) &= \frac{1}{x} \end{aligned}$$

Domain: The set D of all numbers for which $f(x)$ is defined is called the domain of the function f.

For example, consider the function $f(x) = 4x^2 + 5$.

There's no x value where $f(x)$ is undefined so its domain is $\mathbb{R}$.

Range: The set of all values of $f(x)$ is called the range of f.

$$\begin{aligned} x^2 &\geqslant 0 \\ 4x^2 &\geqslant 0 \\ 4x^2+5 &\geqslant 5 \end{aligned}$$

So range of $f(x) = 4x^2 + 5$ is: $\big[5, \infty\big)$.

The range and domain of a linear function $f(x) = ax + b$ is $\mathbb{R}$.

Example 1–10: Find the domain of the function

$$f(x) = \frac{1}{x+8}$$

Solution: Division by zero is undefined, in other words, it is not possible to evaluate this function at the point $x = -8$.

Therefore the domain is: $\mathbb{R} \setminus \{-8\}$.

We can also write this as: $\big(-\infty, -8\big) \cup \big(-8, \infty\big)$

Example 1–11: Find the domain of the function

$$f(x) = \sqrt{x-4}$$

Solution: Square root of a negative number is undefined. (In this course, we are not using complex numbers.) Therefore

$$x - 4 \geqslant 0 \quad \Rightarrow \quad x \geqslant 4$$

In other words, domain is: $\big[4, \infty\big)$

Example 1–12: Find the domain of the function

$$f(x) = \frac{1}{\sqrt{x-4}}$$

Solution: This is similar to previous exercise, but the function is not defined at $x = 4$. Therefore, the domain is: $\big(4, \infty\big)$

EXERCISES

Perform the following operations. Transform and simplify the result.

1–1) $(2^3)^2$

1–2) $\left(\dfrac{1}{16}\right)^{3/4}$

1–3) $72^{1/2}$

1–4) $\sqrt[3]{-125}$

1–5) $\sqrt[3]{\dfrac{8}{1000}}$

1–6) $\sqrt{\dfrac{48}{49}}$

1–7) $(a+b)^2$

1–8) $(a+b)(a-b)$

1–9) $\dfrac{1}{\sqrt{5}-\sqrt{3}}$

1–10) $\dfrac{12}{\sqrt{7}-1} - \dfrac{12}{\sqrt{7}+1}$

Perform the following operations. Transform and simplify the result.

1–11) $\sqrt{x^2\sqrt{x}}$

1–12) $\sqrt{x^3y}\,\sqrt{64xy^9}$

1–13) x^3-1

1–14) $\left(\sqrt{x^2+4}+3\right)\left(\sqrt{x^2+4}-3\right)$

1–15) x^4-100y^4

1–16) $\left(\dfrac{x^2\,y^{1/2}}{x^{2/3}\,y^{1/6}}\right)^3$

1–17) $(3a-2b)^2$

1–18) $(a+b)^3$

1–19) $\dfrac{2x}{x^2-4}+\dfrac{5}{x+2}$

1–20) $1-\dfrac{1}{1+\dfrac{1}{x}}$

Solve the following equations and inequalities:

1–21) $3(x+7)-2(3x-4)=14$

1–22) $\dfrac{x}{3}-\dfrac{x}{5}=\dfrac{7}{30}$

1–23) $\sqrt{x^2+16}=5$

1–24) $|x-2|=12$

1–25) $\left|x-7\right|<8$

1–26) $\left|2x+6\right|\leqslant 4$

1–27) $\left|5x-10\right|>15$

1–28) $\left|12-7x\right|\geqslant 1$

1–29) $\left|x^2-5\right|<2$

1–30) $\left|x^2-5\right|<10$

Find the equations of the following lines:

1–31) Passes through origin and has slope $m=\dfrac{1}{5}$.

1–32) Passes through the point $(-2,6)$ and has slope $m=3$.

1–33) Passes through the points $(-8,2)$ and $(-1,-2)$.

1–34) Passes through $(0,-3)$ and parallel to the line $10y-5x=99$.

1–35) Passes through $(9,12)$ and perpendicular to the line $2x+5y=60$.

Find the domain and range of the following functions:

1–36) $f(x)=\sqrt{10-x}$

1–37) $f(x)=x^2+12x+35$

1–38) $f(x)=8x-x^2$

1–39) $f(x)=\dfrac{1}{x^2-6x+9}$

1–40) $f(x)=\dfrac{3}{x-7}$

ANSWERS

1–1) $2^3 \cdot 2^3 = 2^6 = 64$

1–2) $\left(2^{-4}\right)^{3/4} = 2^{-3} = \dfrac{1}{8}$

1–3) $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

1–4) $\left[(-5)^3\right]^{1/3} = -5$

1–5) $\dfrac{\sqrt[3]{8}}{\sqrt[3]{1000}} = \dfrac{2}{10} = 0.2$

1–6) $\dfrac{4\sqrt{3}}{7}$

1–7) $a^2 + 2ab + b^2$

1–8) $a^2 - b^2$

1–9) $\dfrac{\sqrt{5}+\sqrt{3}}{2}$

1–10) 4

1–11) $x^{5/4}$

1–12) $8x^2y^5$

1–13) $(x-1)(x^2+x+1)$

1–14) $x^2 - 5$

1–15) $(x^2 - 10y^2)(x^2 + 10y^2)$

1–16) x^4y

1–17) $9a^2 - 12ab + 4b^2$

1–18) $a^3 + 3a^2b + 3ab^2 + b^3$

1–19) $\dfrac{7x-10}{x^2-4}$

1–20) $\dfrac{1}{x+1}$

1–21) $x = 5$

1–22) $x = \dfrac{7}{4}$

1–23) $x = \pm 3$

1–24) $x = 14$ or $x = -10$

1–25) $-1 < x < 15$

1–26) $-5 \leqslant x \leqslant -1$

1–27) $x < -1$ or $x > 5$

1–28) $x \leqslant \dfrac{11}{7}$ or $x \geqslant \dfrac{13}{7}$

1–29) $\sqrt{3} < x < \sqrt{7}$ or $-\sqrt{7} < x < -\sqrt{3}$

1–30) $-\sqrt{15} < x < \sqrt{15}$

1–31) $y = \dfrac{1}{5}x$

1–32) $y = 3x + 12$

1–33) $4x + 7y + 18 = 0$

1–34) $x - 2y = 6$

1–35) $5x - 2y = 21$

1–36) Domain: $\big(-\infty, 10\big]$, range: $\big[0, \infty\big)$.

1–37) Domain: $\mathbb{R}$, range: $\big[-1, \infty\big)$.

1–38) Domain: $\mathbb{R}$, range: $\big(-\infty, 16\big)$.

1–39) Domain: $\mathbb{R} \setminus \{3\}$, range: $\big(0, \infty\big)$.

1–40) Domain: $\mathbb{R} \setminus \{7\}$, range: $\big(-\infty, 0\big) \cup \big(0, \infty\big)$.

Chapter 2

Parabolas

Quadratic Equations: The solution of the equation

$$ax^2 + bx + c = 0$$

is:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{where} \quad \Delta = b^2 - 4ac$$

Here, we assume $a \neq 0$.

- If $\Delta > 0$, there are two distinct solutions.
- If $\Delta = 0$, there is a single solution.
- If $\Delta < 0$, there is no real solution.

(In this course, we only consider real numbers)

Example 2–1: Solve the equation $x^2 - 6x - 7 = 0$.

Solution: We can factor this equation as: $(x-7)(x+1) = 0$

Therefore $x - 7 = 0$ or $x + 1 = 0$.

In other words, $x = 7$ or $x = -1$.

Alternatively, we can use the formula to obtain the same result. Note that

$$a = 1,\ b = -6 \quad \text{and} \quad c = -7$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{6 \pm \sqrt{36 + 28}}{2} \\ &= \frac{6 \pm 8}{2} \end{aligned}$$

So $x = 7$ or $x = -1$.

Example 2–2: Solve $8x^2 - 6x - 5 = 0$.

Solution: Using the formula, we obtain:

$$x = \frac{6 \pm \sqrt{36+160}}{16} = \frac{6 \pm 14}{16}$$

So $x = \frac{5}{4}$ or $x = -\frac{1}{2}$.

Alternatively, we can see directly that

$(4x-5)(2x+1) = 0$, but this is not easy.

Example 2–3: Solve $9x^2 - 12x + 4 = 0$.

Solution: If we can see that this is a full square

$(3x-2)^2 = 0$ we obtain $x = \frac{2}{3}$ easily.

Alternatively, $\Delta = (-12)^2 - 4 \cdot 9 \cdot 4 = 0$

(There is only one solution)

Example 2–4: Solve $3x^2 + 6x + 4 = 0$.

Solution:
$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 36 - 48 \\ &= -12 \end{aligned}$$

$\Delta < 0 \quad \Rightarrow \quad$ There is no solution.

Quadratic Functions:

A function of the form

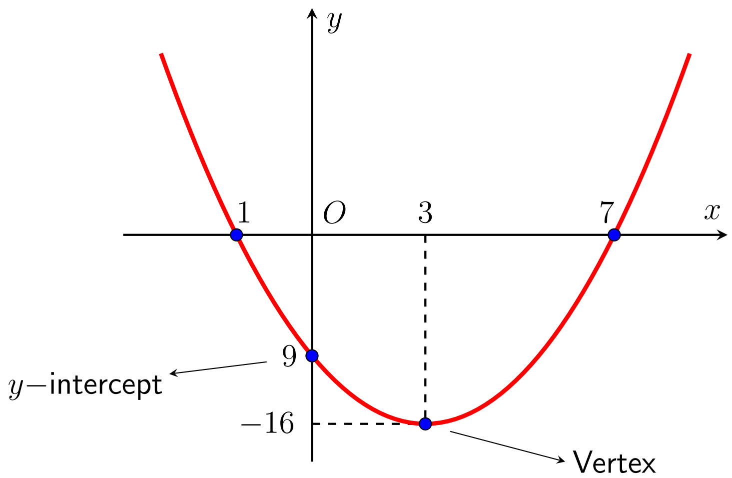
$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a quadratic function. The graph of a quadratic function is a parabola.

- If $a > 0$, the arms of the parabola open upward.
- If $a < 0$, the arms of the parabola open downward.

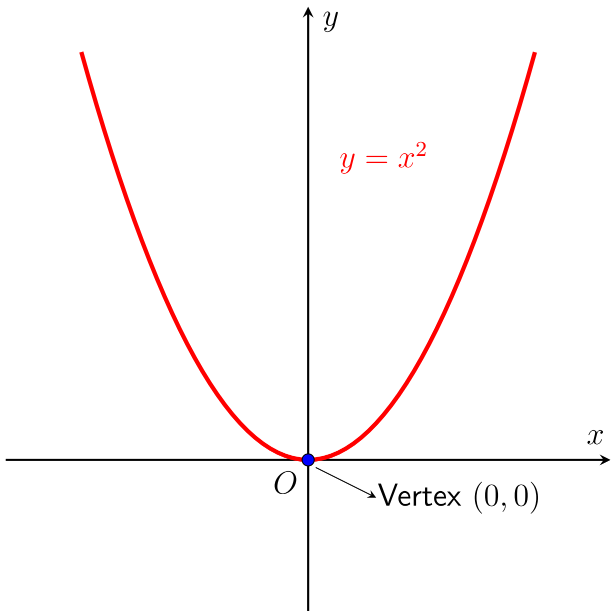
The vertex of the parabola is maximum or minimum point.

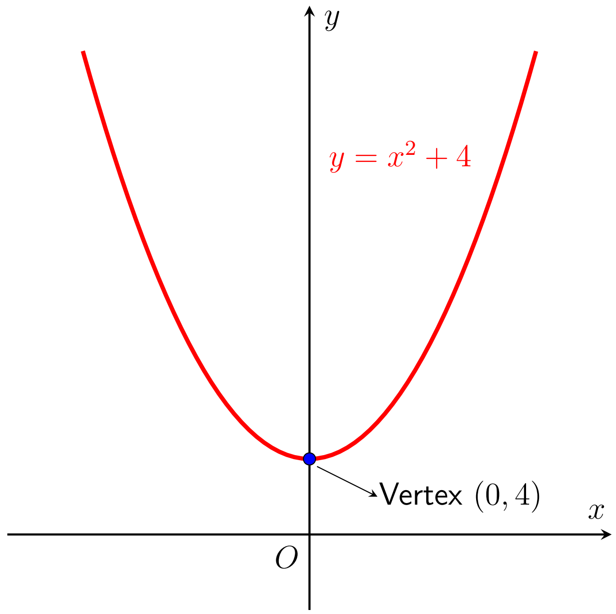
The $x-$coordinate of the vertex is $-\frac{b}{2a}$ and the $y-$coordinate is $f\left(-\frac{b}{2a}\right)$. An example is:

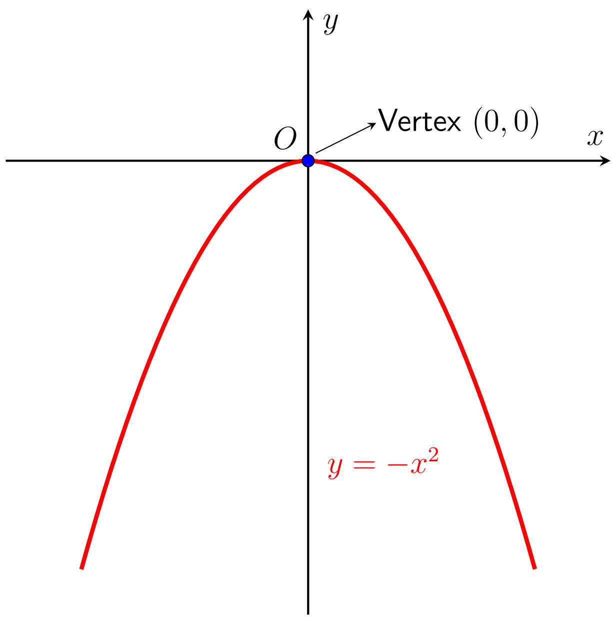


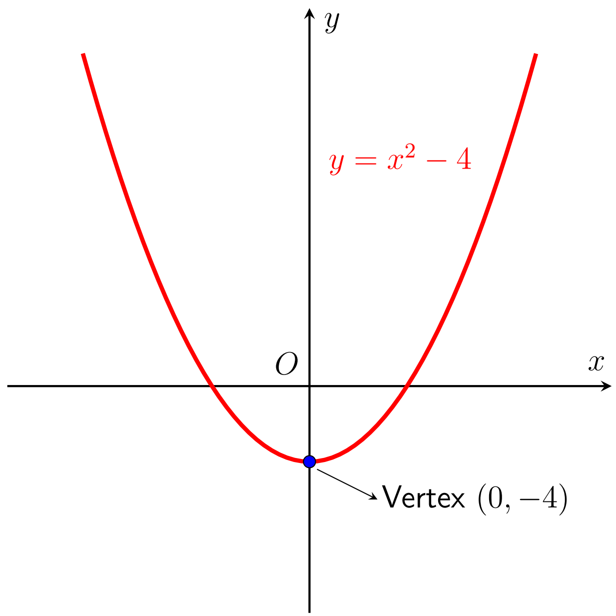
The graph of $y = x^2 - 6x - 7$

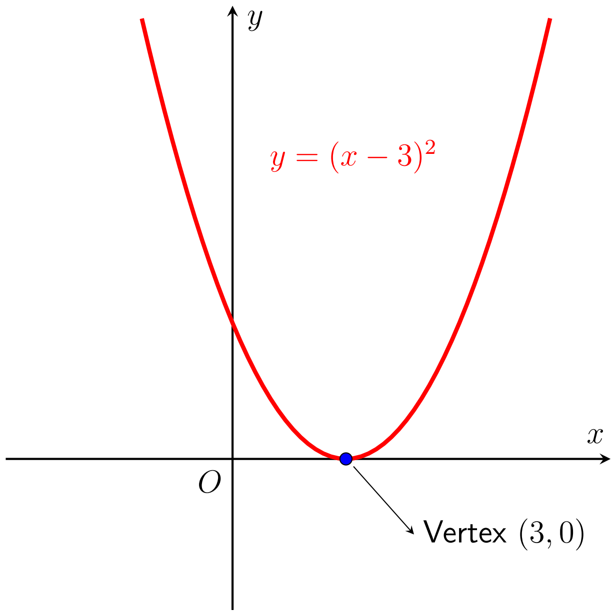
Examples of some parabolas and their vertices:

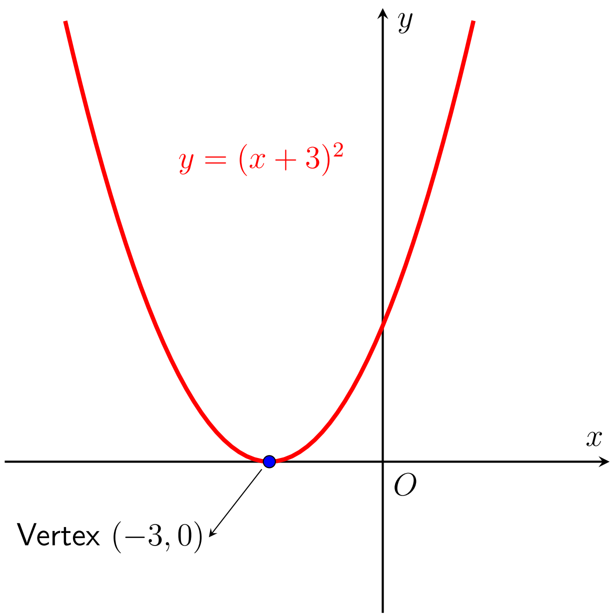












Example 2–5: Sketch the graph of $f(x) = x^2 - 10x + 16$.

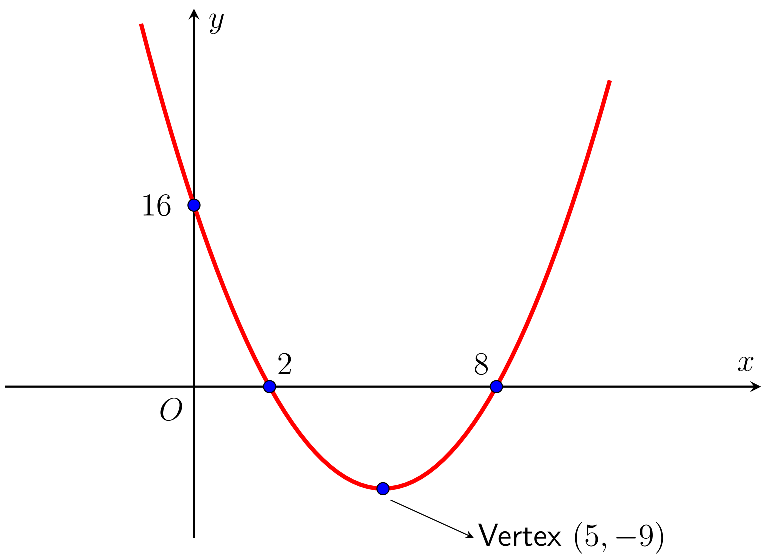
Solution: $y-$intercept: $x = 0 \quad \Rightarrow \quad y = 16$

Roots: $x^2 - 10x + 16 = 0 \quad \Rightarrow \quad x = 2$ or $x = 8$.

Vertex: $-\dfrac{b}{2a} = \dfrac{10}{2} = 5, \quad f(5) = -9.$

The coordinates of the vertex is $(5, -9)$.

$a > 0 \quad \Rightarrow \quad$ arms open upward. The graph is:



We can obtain the same graph by writing the given function in the form:

$$f(x) = (x-5)^2 - 9$$

EXERCISES

Solve the following quadratic equations:

2–1) $x^2 - 5x - 24 = 0$

2–2) $2x^2 + 9x - 5 = 0$

2–3) $6x^2 - 7x + 2 = 0$

2–4) $49x^2 - 14x + 1 = 0$

2–5) $4x^2 + 6x + 3 = 0$

2–6) $x^2 - 17x = 0$

2–7) $4x^2 - 20x + 25 = 0$

2–8) $x^2 - 4x + 5 = 0$

2–9) $x^2 - \dfrac{10}{3}x + 1 = 0$

2–10) $x^2 - 2x - 1 = 0$

Find the vertex and $x-$ and $y-$ intercepts of the following parabolas. Sketch their graphs:

2–11) $y = x^2 - 6x$

2–12) $y = -x^2 + 12$

2–13) $y = x^2 - 4x - 21$

2–14) $y = -x^2 + 3x + 4$

2–15) $y = x^2 + 10x + 25$

2–16) $y = 4x^2 - 8x + 3$

2–17) $y = 5x^2 + 15$

2–18) $y = -(x - 4)^2$

2–19) $y = x^2 - 4x + 5$

2–20) $y = -3x^2 + 60x - 450$

ANSWERS

2–1) $x_1 = 8, \quad x_2 = -3.$

2–2) $x_1 = \frac{1}{2}, \quad x_2 = -5.$

2–3) $x_1 = \frac{1}{2}, \quad x_2 = \frac{2}{3}.$

2–4) $x_1 = \frac{1}{7}$. (double root.)

2–5) There is no solution.

2–6) $x_1 = 0, \quad x_2 = 17.$

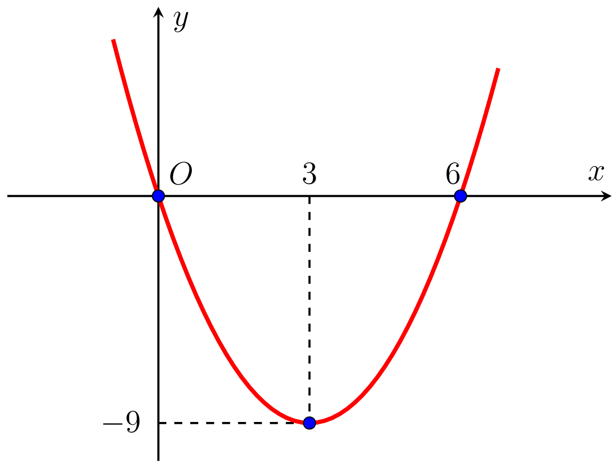
2–7) $x_1 = \frac{5}{2}$. (double root.)

2–8) There is no solution.

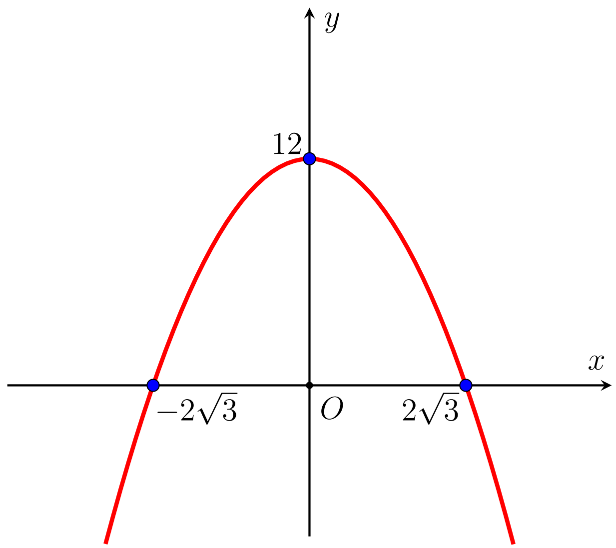
2–9) $x_1 = 3, \quad x_2 = \frac{1}{3}.$

2–10) $x_1 = 1 + \sqrt{2}, \quad x_2 = 1 - \sqrt{2}.$

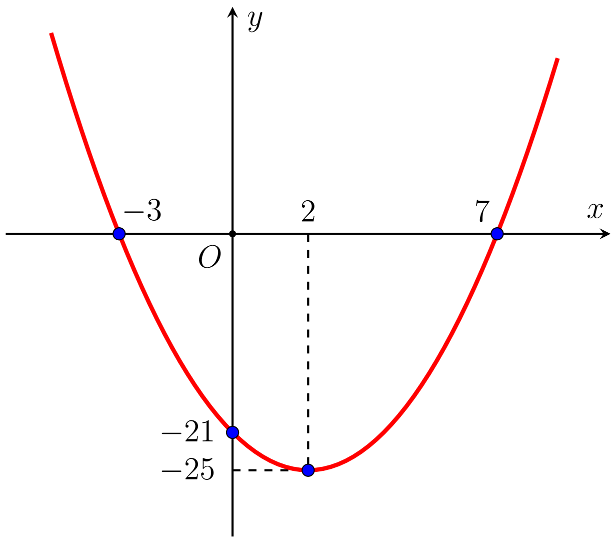
2–11)



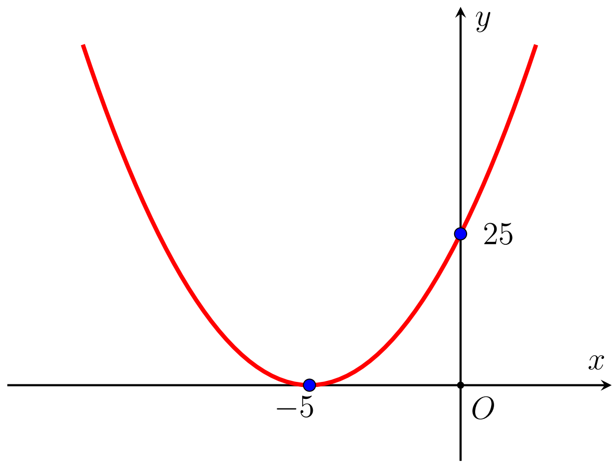
2–12)



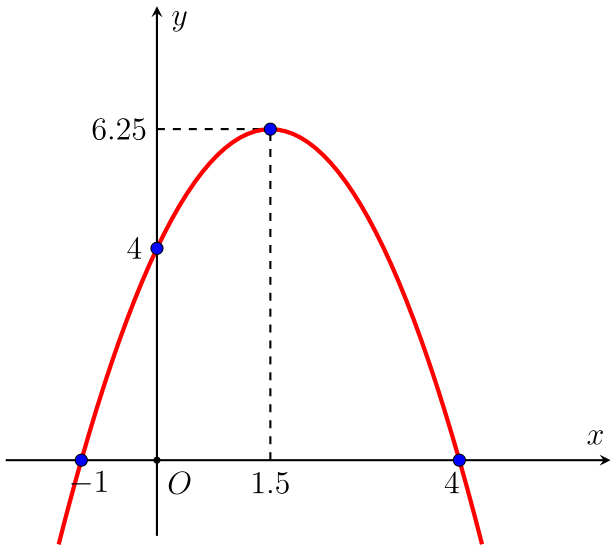
2–13)



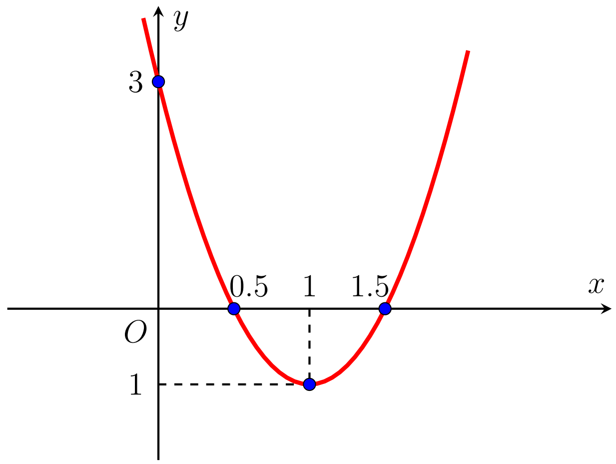
2–15)



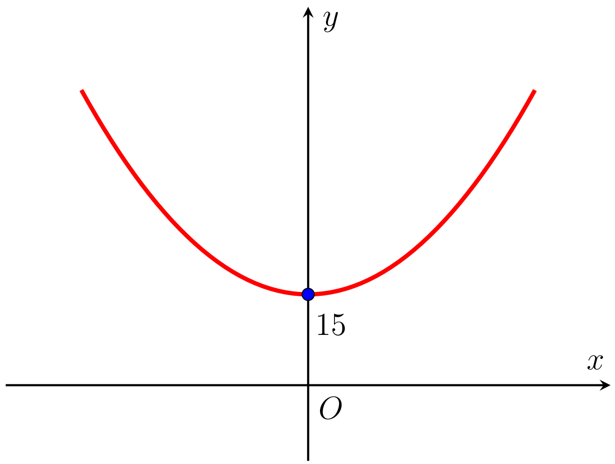
2–14)



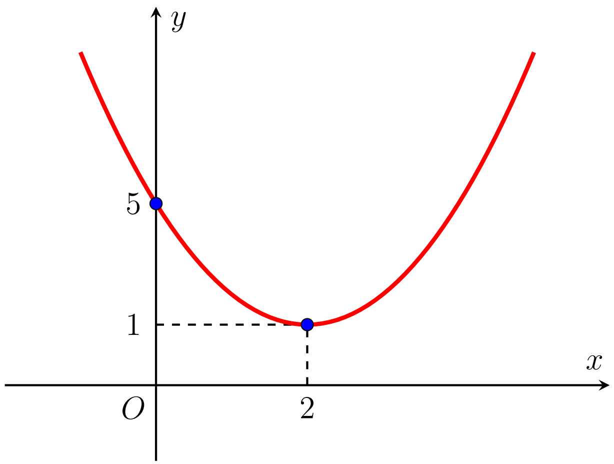
2–16)



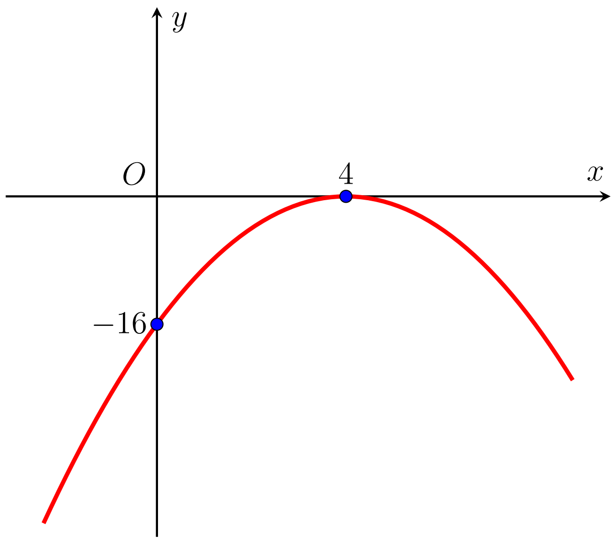
2–17)



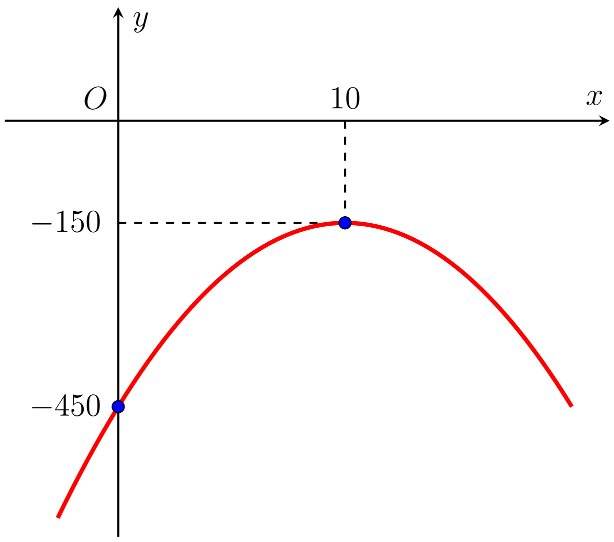
2–19)



2–18)



2–20)



Chapter 3

Exponential and Logarithmic Functions

Polynomials: A function of the form

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a polynomial of degree n. For example, $120x^5 - 17x + \dfrac{7}{2}$ is a polynomial.

$\sqrt{x}, \quad x^{-1}, \quad \dfrac{1}{1+x}, \quad x^{5/3}$ are NOT polynomials.

Rational Functions: The quotient of two polynomials is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. For example,

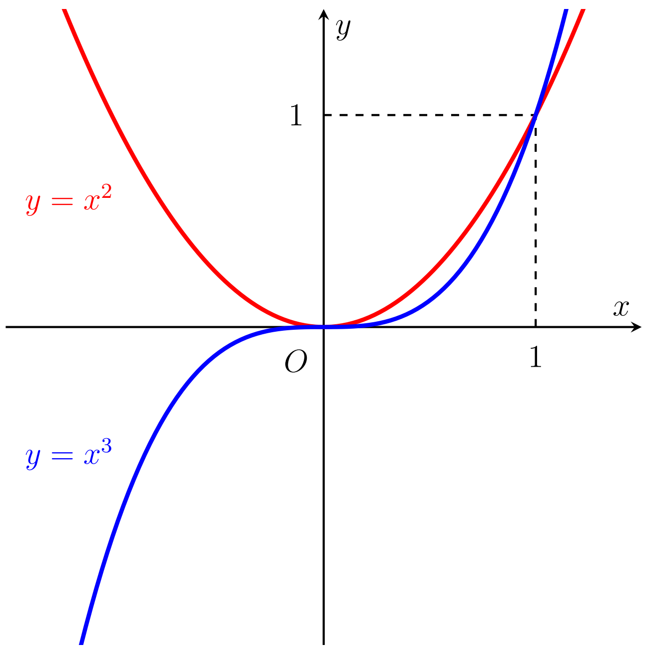
$$\frac{3x^2 - 5}{1 + 2x - 7x^3}$$

is a rational function.

Question: What is the domain of a polynomial? A rational function?

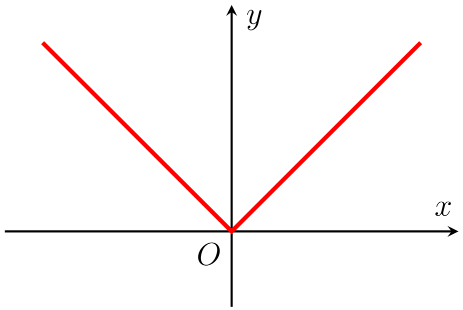
Example 3–1: Sketch the functions $y = x^2$ and $y = x^3$ on the same coordinate system.

Solution:

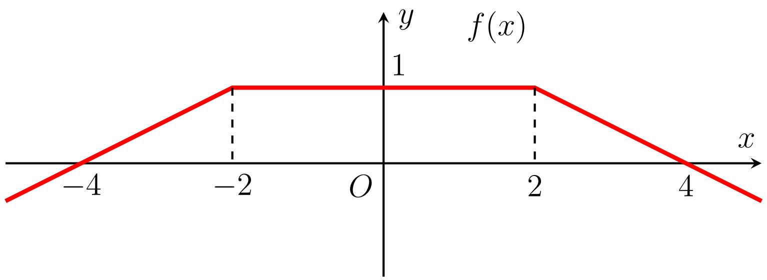


Piecewise-Defined Functions: We may define a function using different formulas for different parts of the domain. For example, the absolute value function is:

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$



Example 3–2: Find the formula of the function $f(x)$:



Solution:

$$f(x) = \begin{cases} \dfrac{x+4}{2} & \text{if} \quad x < -2 \\ 1 & \text{if} \quad -2 \leqslant x \leqslant 2 \\ \dfrac{-x+4}{2} & \text{if} \quad x > 2 \end{cases}$$

Inverse Functions:
If $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$, the functions f and g are inverses of each other.

For example, the inverse of $f(x) = 2x + 1$ is:

$$f^{-1}(x) = g(x) = \frac{x-1}{2}$$

One–to–one Functions: If $f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$ then f is one-to one.

For example, $f(x) = x^3$ is one-to one but $g(x) = x^2$ is not, because $g(1) = g(-1)$.

Onto Functions: Let $f : A \to B$. If there exists an $x \in A$ for all $y \in B$ such that $f(x) = y$ then f is onto.

For example, $f(x) = 2x + 1$ is onto but $g(x) = |x|$ is not, because there is no x such that $g(x) = -2$ or any other negative number.

Theorem: A function has an inverse if and only if it is one-to-one and onto.

Example 3–3: Find the inverse of the function $f(x) = \dfrac{x-2}{x+1}$ on the domain $\mathbb{R} \setminus \{-1\}$ and range $\mathbb{R} \setminus \{1\}$.

Solution: $y = \dfrac{x-2}{x+1} \quad \Rightarrow \quad yx + y = x - 2$

$$yx - x = -y - 2 \quad \Rightarrow \quad x(y-1) = -y - 2$$

$$x = -\frac{y+2}{y-1}$$

In other words, $f^{-1}(x) = -\dfrac{x+2}{x-1}$.

Exponential Functions: Functions of the form

$$f(x) = a^x$$

where a is a positive constant (but $a \neq 1$) are called exponential functions. The domain is:

$$\mathbb{R} = (-\infty, \infty)$$

and the range is

$$(0, \infty)$$

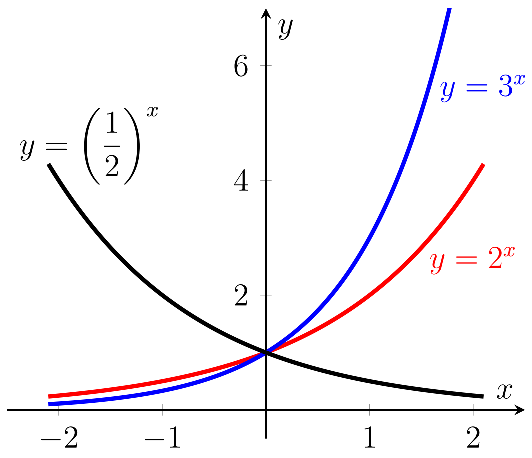
Remember that:

- $a^n = a \cdot a \cdots a$
- $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$
- $a^{1/n} = \sqrt[n]{a}$
- $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

The natural exponential function is:

$$f(x) = e^x$$

where $e = 2.71828\ldots$.



For the exponential function $f(x) = 2^x$,

$$\begin{aligned} f(6) &= 64 \\ f(5) &= 32 \\ f(1) &= 2 \\ f(0) &= 1 \\ f\left(\frac{1}{2}\right) &= \sqrt{2} \end{aligned}$$

Do not confuse this with the polynomial function $g(x) = x^2$ because

$$\begin{aligned} g(6) &= 36 \\ g(5) &= 25 \\ g(1) &= 1 \\ g(0) &= 0 \\ g\left(\frac{1}{2}\right) &= \frac{1}{4} \end{aligned}$$

Example 3–4: If we invest an amount A in the bank, and if the rate of interest is 15% per year, how much money will we have after n years?

Solution: We are multiplying by 1.15 every year, so: $1.15^n A$.

Example 3–5: A firm has C customers now. Every month, 30% of the customers leave. How many remain after n months?

Solution: We are multiplying by 0.7 every month, so: $0.7^n C$.

Logarithmic Functions: The inverse of the exponential function $y = a^x$ is the logarithmic function with base a:

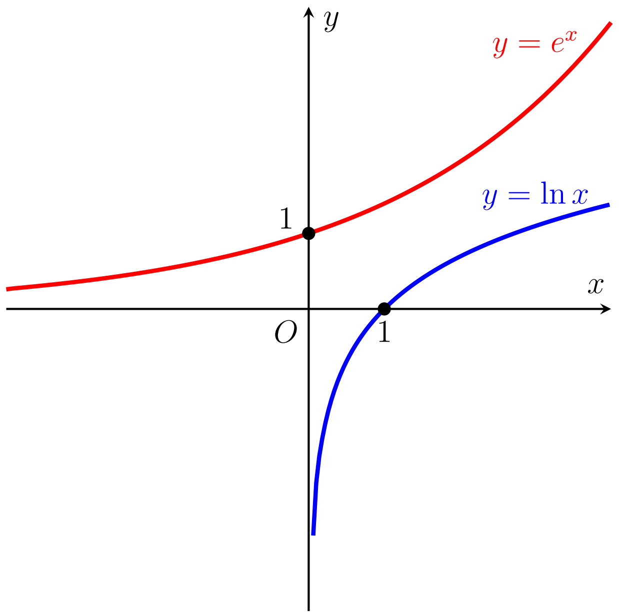
$$y = \log_a x$$

where $a > 0, \quad a \neq 1$.

$$a^{\log_a x} = \log_a(a^x) = x$$

We will use:

- $\log x$ for $\log_{10} x$ (common logarithm)
- $\ln x$ for $\log_e x$ (natural logarithm)



We can easily see that,

$$a^x \cdot a^y = a^{x+y} \quad \Rightarrow \quad \log_a(AB) = \log_a A + \log_a B$$

As a result of this,

- $\log_a \left(\dfrac{A}{B}\right) = \log_a A - \log_a B$
- $\log_a \left(\dfrac{1}{B}\right) = -\log_a B$
- $\log_a (A^r) = r \log_a A$

Any logarithm can be expressed in terms of the natural logarithm:

$$\log_a(x) = \frac{\ln x}{\ln a}$$

Any exponential can be expressed in terms of the natural exponential:

$$a^x = e^{x \ln a}$$

Example 3–6: Simplify $\log 360$.

Solution: First, we have to find factors of 360:

$360 = 2^3 \cdot 3^2 \cdot 5$

Now, we can use the properties of logarithms:

$$\begin{aligned} \log 360 &= \log 2^3 + \log 3^2 + \log 5 \\ &= 3\log 2 + 2\log 3 + 1 - \log 2 \\ &= 1 + 2\log 2 + 2\log 3 \end{aligned}$$

EXERCISES

Sketch the graphs of the following piecewise-defined functions:

3–1) $f(x) = \begin{cases} 2x & \text{if} \quad x < 5 \\ 10 & \text{if} \quad x \geqslant 5 \end{cases}$

3–2) $f(x) = \begin{cases} x+3 & \text{if} \quad x < 4 \\ x-1 & \text{if} \quad x \geqslant 4 \end{cases}$

Are the following functions polynomials?

3–3) $f(x) = 8x^4 + 1$

3–4) $f(x) = \dfrac{1-x}{x}$

3–5) $f(x) = \dfrac{1}{5}x + \dfrac{1}{3}$

3–6) $f(x) = 5x^5 - 3x^{2/3}$

Are the following functions one-to-one? Are they onto?

3–7) $f(x) = 2x$

3–8) $f(x) = x^3$

3–9) $f(x) = x^4 + x^2 + 1$

3–10) $f(x) = e^{2x}$

Find the inverse of the following functions.

3–11) $f(x) = 3x - 2$

3–12) $f(x) = \dfrac{x+2}{5x+4}$

3–13) $f(x) = \dfrac{1}{x}$

3–14) $f(x) = x^3 + 1$

Simplify the following:

3–15) $\log 400$

3–16) $\log 288$

3–17) $\log_9 27$

3–18) $\log_8 16$

3–19) $\log_2 1250$

3–20) $\log_3 \dfrac{\sqrt{3}}{81}$

3–21) $e^{2x+5\ln x}$

3–22) $\ln \dfrac{e}{\sqrt[3]{e}}$

3–23) $2^{3x+4\log_2 x}$

3–24) $3^{2\log_9 x}$

3–25) $5^{\log_{25} x}$

3–26) $10^{1+\log(2x)}$

Solve the following equations.

3–27) $5 = \left(5\sqrt{5}\right)^x$

3–28) $\log_x 12 = \dfrac{1}{2}$

3–29) $\log_x 77 = -1$

3–30) $\log_x 2 = 3$

3–31) $\log_x 64 = 4$

3–32) $\log_3 x = 5$

3–33) $\log_9(18x) = 2$

3–34) $\log_5 x = -\dfrac{1}{2}$

3–35) $\log(\log x) = 0$

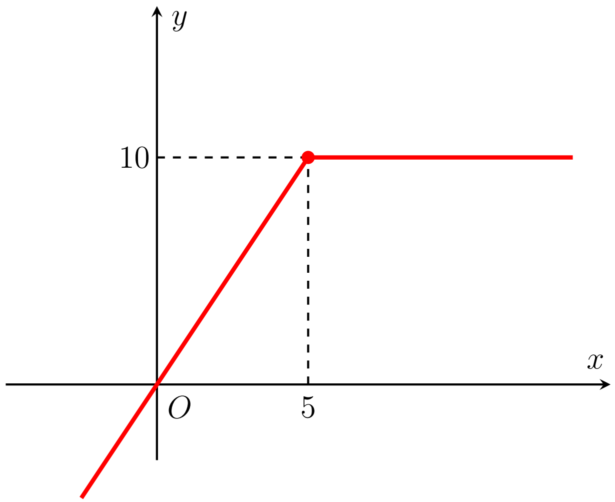
3–36) $\ln(\ln x) = 1$

3–37) $2^x = 100$

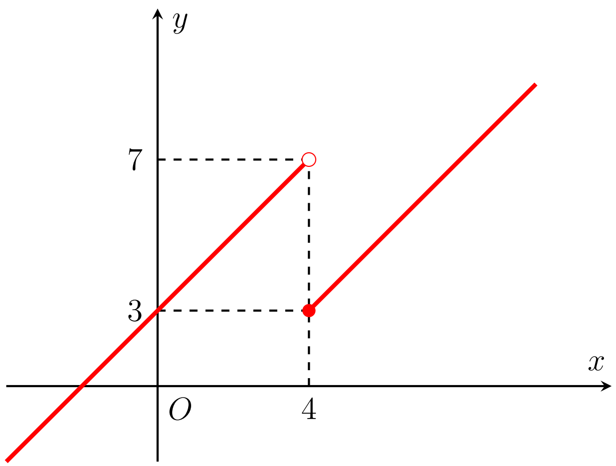
3–38) $2^{4x+4} = 8^{x-1}$

ANSWERS

3–1)



3–2)



3–3) Yes

3–4) No

3–5) Yes

3–6) No

3–7) One-to-one and onto.

3–8) One-to-one and onto.

3–9) Not one-to-one and not onto.

3–10) One-to-one and not onto.

3–11) $f^{-1}(x) = \dfrac{x+2}{3}$

3–12) $f^{-1}(x) = \dfrac{4x-2}{1-5x}$

3–13) $f^{-1}(x) = \dfrac{1}{x}$

3–14) $f^{-1}(x) = \sqrt[3]{x-1}$

3–15) $2 + 2\log 2$

3–16) $2\log 3 + 5\log 2$

3–17) $\dfrac{3}{2}$

3–18) $\dfrac{4}{3}$

3–19) $1 + 4\log_2 5$

3–20) $-\dfrac{7}{2}$

3–21) $x^5 e^{2x}$

3–22) $\dfrac{2}{3}$

3–23) $x^4 8^x$

3–24) x

3–25) $\sqrt{x}$

3–26) $20x$

3–27) $x = \dfrac{2}{3}$

3–28) $x = 144$

3–29) $x = \dfrac{1}{77}$

3–30) $x = 2^{1/3}$

3–31) $x = 2\sqrt{2}$

3–32) $x = 243$

3–33) $x = \dfrac{9}{2}$

3–34) $x = \dfrac{1}{\sqrt{5}}$

3–35) $x = 10$

3–36) $x = e^e$

3–37) $x = \dfrac{2}{\log 2}$

3–38) $x = -7$

Chapter 26

Matrices and Basic Operations

An $n \times m$ matrix is a rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

This matrix has n **rows** and m **columns**. The numbers a_{ij} are called **entries**.

If $m = n$, it is called a **square matrix**, for example

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 9 & 8 & 7 \end{bmatrix}$$

is a square matrix. The **main diagonal** entries of a square matrix are the entries $a_{11}, a_{22}, \ldots, a_{nn}$.

$$A = \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{2} & 4 \\ 9 & 8 & \boxed{7} \end{bmatrix}$$

A **diagonal matrix** is a square matrix where all the entries that are not on the main diagonal are zero, for example

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

An **upper triangular** matrix is a square matrix where all elements below diagonal are zero, and a **lower triangular** matrix is a square matrix where all elements above diagonal are zero, for example:

$$C = \begin{bmatrix} 6 & 4 & 2 & 11 \\ 0 & 2 & 18 & 20 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 15 & 7 & 0 & 0 \\ 6 & 33 & 8 & 0 \\ -5 & 13 & 40 & 1 \end{bmatrix}$$

The **transpose** of a matrix A is A^T, obtained by interchanging rows and columns, for example:

$$E = \begin{bmatrix} 3 & 5 & 7 \\ 10 & 2 & 4 \\ -8 & 0 & 9 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 3 & 10 & -8 \\ 5 & 2 & 0 \\ 7 & 4 & 9 \end{bmatrix}$$

are transposes of each other. $F = E^T$ and $E = F^T$

If $A = A^T$, we call it a **symmetric matrix**, for example:

$$A = \begin{bmatrix} 1 & 2 & 9 \\ 2 & 4 & 7 \\ 9 & 7 & 5 \end{bmatrix}$$

We can add and subtract matrices by adding or subtracting their corresponding entries, for example:

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 8 & 22 \end{bmatrix} + \begin{bmatrix} 12 & -3 & 5 \\ 9 & 1 & 11 \end{bmatrix} = \begin{bmatrix} 13 & -3 & 3 \\ 11 & 9 & 33 \end{bmatrix}$$

If matrix dimensions are different, we can not add or subtract them, for example:

$$\begin{bmatrix} 7 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 9 \\ 1 & 2 & 3 \\ 5 & 4 & 0 \end{bmatrix} = ??$$

is undefined.

If A is a matrix and c is a scalar (a number), then $B = cA$ is the **scalar multiplication** of c and A.

$$b_{ij} = c\, a_{ij}$$

In other words, we multiply each entry. For example:

$$7 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 20 & -1 \\ 4 & 5 & 11 \end{bmatrix} = \begin{bmatrix} 7 & 14 & 21 \\ 0 & 140 & -7 \\ 28 & 35 & 77 \end{bmatrix}$$

Example 26–1: Let

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 \\ 11 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} 8 & 3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix}$$

Calculate the following if possible:

a) $3A + B$

b) $A - B^T$

c) $A + C$

d) $C - D$

e) $2C + D^T$

Solution: a) $3A + B = \begin{bmatrix} 0 & 14 \\ 11 & 24 \end{bmatrix}$

b) $A - B^T = \begin{bmatrix} 4 & -7 \\ -2 & -4 \end{bmatrix}$

c) $A + C$ is undefined, dimensions are different.

d) $C - D$ is undefined, dimensions are different.

e) $2C + D^T = \begin{bmatrix} 10 & 10 & 12 \\ 7 & 9 & 12 \end{bmatrix}$

Matrix Multiplication: If A is an $n \times k$ matrix and B is a $k \times m$ matrix, then $C = AB$ is an $n \times m$ matrix. The entries of C are calculated using:

$$c_{ij} = \sum_{p=1}^{k} a_{ip} b_{pj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

For example

$$\begin{bmatrix} 3 & 4 & 8 \\ 1 & 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 10 & 7 \end{bmatrix}$$

is impossible, because number of columns of the first matrix and the number of rows of the second matrix are not equal.

$$AB = \begin{bmatrix} 3 & 4 & 8 \\ 1 & 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 & 7 & 14 \\ 10 & 8 & 21 & 1 \\ 0 & -3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}$$

This is possible because A has three columns and B has 3 rows. Some of the entries are:

$$c_{11} = 3 \cdot 5 + 4 \cdot 10 + 8 \cdot 0 = 55$$

$$c_{12} = 3 \cdot 2 + 4 \cdot 8 + 8 \cdot -3 = 14$$

$$c_{21} = 1 \cdot 5 + 7 \cdot 10 + 5 \cdot 0 = 75$$

etc. To find c_{ij}, multiply the i^{th} row of A with j^{th} column of B. For example, to find c_{13}:

$$\begin{bmatrix} \boxed{\bullet} & \boxed{\bullet} & \boxed{\bullet} \\ \bullet & \bullet & \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \boxed{\bullet} & \bullet \\ \bullet & \bullet & \boxed{\bullet} & \bullet \\ \bullet & \bullet & \boxed{\bullet} & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \boxed{\bullet} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

Example 26–2: Let $A = \begin{bmatrix} 1 & 7 \\ 8 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 9 & 0 \end{bmatrix}$,

$$C = \begin{bmatrix} 1 & 6 & 5 \\ 0 & 4 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 7 & -2 & 3 \\ -1 & 2 & 4 \end{bmatrix}.$$

Find the following matrix products (if possible).

a) AB
b) BA
c) CD
d) CD^T
e) $C^T D$
f) DB

Solution: **a)** $\begin{bmatrix} 1 & 7 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} 66 & 5 \\ 42 & 40 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 43 & 31 \\ 9 & 63 \end{bmatrix}$

c) Undefined

d) $\begin{bmatrix} 1 & 6 & 5 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ -2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 31 \\ -2 & 16 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 \\ 6 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 7 & -2 & 3 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -2 & 3 \\ 38 & -4 & 34 \\ 33 & -6 & 23 \end{bmatrix}$

f) Undefined

Matrix product is associative:

$$A(BC) = (AB)C$$

For example, consider the product

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 7 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix}$$

We can find the result in two different ways:

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \left(\begin{bmatrix} 3 & -1 \\ 7 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix} \right) &= \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 19 & 14 \\ 35 & 42 \end{bmatrix} \\ &= \begin{bmatrix} 89 & 98 \\ 140 & 168 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 7 & 0 \end{bmatrix} \right) \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix} &= \begin{bmatrix} 17 & -1 \\ 28 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 89 & 98 \\ 140 & 168 \end{bmatrix} \end{aligned}$$

The results are the same.

But matrix product is not commutative:

$$AB \neq BA \quad \text{(In general)}$$

The transpose of the product is product of transposes in reverse order:

$$(AB)^T = B^T A^T$$

An **identity matrix** is a diagonal matrix whose all nonzero entries are 1. We denote identity matrices by I_n or simply I.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product of an identity matrix with any other matrix A gives the result A. (when the product is defined.) For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 9 \\ 5 & 7 & 11 \end{bmatrix}$$

For a square matrix A, we have:

- $A^n = AA \cdots A \quad (n \text{ times})$
- $A^n A^m = A^{n+m}$
- $(A^n)^m = A^{nm}$
- $A^0 = I$

The powers of a diagonal matrix are especially easy to calculate, for example:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}^8 = \begin{bmatrix} 3^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EXERCISES

The matrices A, B, C, D are defined as:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 1 & -5 \\ -1 & -2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 1 \\ 0 & 6 \\ 3 & 0 \end{bmatrix}.$$

Perform the following operations if possible:

26–1) $A + 4B$

26–2) AB

26–3) CD

26–4) DC

26–5) $C + D$

26–6) ADC

26–7) DAC

26–8) $2BA - 5DC$

26–9) $CB - AD$

26–10) $AD - BD$

26–11) $A + B^T$

26–12) $\left(A + B\right)^T$

26–13) $\left(A^T - 2B\right)^T$

26–14) $C^T + D$

26–15) $D^T - 2C^T$

26–16) $B^2 - 2B - I$

Let $A = \begin{bmatrix} 2 & -1 \\ 5 & 7 \end{bmatrix}$,

$B = \begin{bmatrix} 3 & 4 & 0 \\ 8 & 5 & 7 \end{bmatrix}$,

$C = \begin{bmatrix} -2 & 9 & 1 \\ 4 & 6 & -5 \\ 3 & 2 & 0 \end{bmatrix}$,

$D = \begin{bmatrix} 6 & 2 \\ 0 & -2 \\ 5 & 1 \end{bmatrix}$.

Calculate the following if possible:

26–17) AB^T

26–18) BC

26–19) DC

26–20) $(BD)^T$

26–21) DB

26–22) CD

Evaluate the following matrix products if they are defined:

26–23) $\begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 10 & 7 & 9 \\ 0 & 1 & 4 \end{bmatrix}$

26–24) $\begin{bmatrix} 10 & 7 & 9 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix}$

26–25) $\begin{bmatrix} 2 & 0 & 3 & -1 \\ 8 & 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 2 & 2 \\ 6 & 5 \\ 9 & -2 \end{bmatrix}$

26–26) $\begin{bmatrix} 7 & 0 \\ 2 & 2 \\ 6 & 5 \\ 9 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 & -1 \\ 8 & 5 & 0 & 4 \end{bmatrix}$

26–27) $\begin{bmatrix} 4 & -3 & 8 \\ 9 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \\ 2 & 4 & 12 \\ 0 & 8 & 0 \end{bmatrix}$

26–28) $\begin{bmatrix} 9 & 5 & 1 \\ 4 & 0 & 8 \\ 7 & 2 & 11 \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 6 & 8 & 1 \\ -4 & 2 & -1 \end{bmatrix}$

26–29) $\begin{bmatrix} 3 & 0 & -3 \\ 6 & 8 & 1 \\ -4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & 5 & 1 \\ 4 & 0 & 8 \\ 7 & 2 & 11 \end{bmatrix}$

26–30) $\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 8 & 10 & 12 & 16 \\ 2 & 3 & 5 & 3 \end{bmatrix}$

ANSWERS

26–1) $\begin{bmatrix} 33 & 6 & 0 \\ 11 & 3 & 6 \\ 4 & 13 & 9 \end{bmatrix}$

26–2) $\begin{bmatrix} 14 & 1 & 2 \\ 1 & 3 & 11 \\ 47 & -10 & -23 \end{bmatrix}$

26–3) $\begin{bmatrix} -7 & 8 \\ -4 & -13 \end{bmatrix}$

26–4) $\begin{bmatrix} 7 & 2 & -20 \\ -6 & -12 & 0 \\ 6 & 3 & -15 \end{bmatrix}$

26–5) Undefined

26–6) $\begin{bmatrix} -5 & -22 & -20 \\ -13 & -32 & -10 \\ -44 & -73 & 25 \end{bmatrix}$

26–7) Undefined

26–8) $\begin{bmatrix} -21 & 28 & 104 \\ 44 & 82 & -14 \\ -2 & 37 & 27 \end{bmatrix}$

26–9) Undefined

26–10) $\begin{bmatrix} -28 & -1 \\ -13 & 14 \\ -17 & 22 \end{bmatrix}$

26–11) $\begin{bmatrix} 9 & 5 & 0 \\ 0 & 3 & 4 \\ 4 & 6 & -3 \end{bmatrix}$

26–12) $\begin{bmatrix} 9 & 2 & 4 \\ 3 & 3 & 7 \\ 0 & 3 & -3 \end{bmatrix}$

26–13) $\begin{bmatrix} -15 & -4 & 0 \\ -3 & 3 & -2 \\ 4 & 3 & -15 \end{bmatrix}$

26–14) $\begin{bmatrix} 6 & 0 \\ 1 & 4 \\ -2 & 0 \end{bmatrix}$

26–15) Undefined

26–16) $\begin{bmatrix} 50 & 6 & 1 \\ 18 & 4 & 2 \\ 6 & 4 & 9 \end{bmatrix}$

26–17) Undefined

26–18) $\begin{bmatrix} 10 & 51 & -17 \\ 25 & 116 & -17 \end{bmatrix}$

26–19) Undefined

26–20) $\begin{bmatrix} 18 & 83 \\ -2 & 13 \end{bmatrix}$

26–21) $\begin{bmatrix} 34 & 34 & 14 \\ -16 & -10 & -14 \\ 23 & 25 & 7 \end{bmatrix}$

26–22) $\begin{bmatrix} -7 & -21 \\ -1 & -9 \\ 18 & 2 \end{bmatrix}$

26–23) $\begin{bmatrix} 10 & 10 & 21 \\ 40 & 37 & 72 \end{bmatrix}$

26–24) Undefined.

26–25) $\begin{bmatrix} 23 & 17 \\ 102 & 2 \end{bmatrix}$

26–26) $\begin{bmatrix} 14 & 0 & 21 & -7 \\ 20 & 10 & 6 & 6 \\ 52 & 25 & 18 & 14 \\ 2 & -10 & 27 & -17 \end{bmatrix}$

26–27) $\begin{bmatrix} -2 & 72 & -8 \\ 9 & 69 & 63 \end{bmatrix}$

26–28) $\begin{bmatrix} 53 & 42 & -23 \\ -20 & 16 & -20 \\ -11 & 38 & -30 \end{bmatrix}$

26–29) $\begin{bmatrix} 6 & 9 & -30 \\ 93 & 32 & 81 \\ -35 & -22 & 1 \end{bmatrix}$

26–30) $\begin{bmatrix} 14 & 17 & 19 & 29 \\ 0 & 2 & 8 & -4 \end{bmatrix}$

Chapter 27

Row Reduction

Row-Echelon Form:
A matrix that satisfies the following conditions is in **row-echelon form**:

- If there are rows consisting of zeros only, they are at the bottom.
- The first nonzero item (from left to right) of each row is 1. It is called a **leading 1**.
- Each leading 1 is to the right of the other leading 1's above it.

If a matrix is in row-echelon form, we can solve the system of equations represented by that matrix easily.
Some examples of matrices in row-echelon form (REF) are:

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 4 \\ 0 & 0 & \boxed{1} \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & -5 & 10 & -1 \\ 0 & \boxed{1} & 9 & 7 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 5 & 9 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 6 & 12 & 4 \\ 0 & \boxed{1} & 3 & 5 \\ 0 & 0 & \boxed{1} & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 23 & 0 & -8 \\ 0 & \boxed{1} & 12 & 7 \\ 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

The following matrices are NOT in REF:

$$\begin{bmatrix} 1 & 5 & 11 & 8 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 44 & 0 & 16 \\ 0 & 1 & 21 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -3 & 2 & 6 & 12 \\ 0 & 1 & 48 & 17 & 21 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 33 & -7 \end{bmatrix}$$

Elementary Row Operations:
There are three types of elementary row operations. We can:

- Interchange two rows,
- Multiply a row by a nonzero constant,
- Add a multiple of a row to another row.

If we do these operations on a matrix, we obtain a **row-equivalent** matrix.

Note that these operations are similar to operations on linear systems of equations, in the sense that they do not change the solution set.

We can reduce matrices to REF using elementary row operations in a systematic way.

It is possible to use different steps to reduce a matrix. For example, if you have a 4 at the top left and 1 below it, you can multiply row 1 by $\frac{1}{4}$ or you can interchange row 1 and row 2.

Example 27–1: Find a REF matrix that is row equivalent to

$$\begin{bmatrix} 0 & 1 & 4 \\ 2 & 4 & -6 \\ 3 & 8 & 0 \end{bmatrix}$$

Solution: Interchange the first two rows:

$$R_1 \longleftrightarrow R_2 \qquad \Longrightarrow \begin{bmatrix} 2 & 4 & -6 \\ 0 & 1 & 4 \\ 3 & 8 & 0 \end{bmatrix}$$

Divide row 1 by two:

$$R_1 \rightarrow \frac{1}{2} R_1 \qquad \Longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 3 & 8 & 0 \end{bmatrix}$$

Multiply row 1 by three and subtract from row 3:

$$R_3 \rightarrow R_3 - 3R_1 \quad \Longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 2 & 9 \end{bmatrix}$$

Multiply row 2 by two and subtract from row 3:

$$R_3 \rightarrow R_3 - 2R_2 \quad \Longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced Row-Echelon Form:
Consider a matrix that is in row-echelon form. If it satisfies the further condition

- If there's a leading 1 in a column, all other entries in that column are zero.

we say that matrix is in **reduced row-echelon form** (RREF).
An identity matrix is a typical RREF matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Some other examples are:

$$\begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 24 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 15 & 0 & 0 & 23 & 36 \\ 0 & 0 & 1 & 0 & 40 & 7 \\ 0 & 0 & 0 & 1 & 54 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 27–2: Using elementary row operations, reduce the following matrix to RREF:

$$\begin{bmatrix} 2 & -6 & -2 & 8 \\ -2 & 11 & 7 & 12 \\ -3 & 10 & 5 & -1 \end{bmatrix}$$

Solution: $R_1 \to \dfrac{1}{2} R_1$ $\qquad \begin{bmatrix} 1 & -3 & -1 & 4 \\ -2 & 11 & 7 & 12 \\ -3 & 10 & 5 & -1 \end{bmatrix}$

$$\begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix} \qquad \begin{bmatrix} 1 & -3 & -1 & 4 \\ 0 & 5 & 5 & 20 \\ 0 & 1 & 2 & 11 \end{bmatrix}$$

$$R_2 \to \frac{1}{5} R_2 \qquad \begin{bmatrix} 1 & -3 & -1 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 2 & 11 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \qquad \begin{bmatrix} 1 & -3 & -1 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

This is REF.

$$\begin{matrix} R_2 \to R_2 - R_3 \\ R_1 \to R_1 + R_3 \end{matrix} \qquad \begin{bmatrix} 1 & -3 & 0 & 11 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$R_1 \to R_1 + 3R_2 \qquad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

This is RREF.

Method of Reduction:

- A leading 1 in first in position a_{11} if possible. If not, next right position (a_{12}).
- All entries below the leading 1 must be zero.
- Similarly, a leading 1 in second row in position a_{22} if possible. If not, next right position (a_{23}).
- All entries below the leading 1 must be zero.
- $\cdots$

At the end of this, we obtain a REF matrix.

If we want a RREF matrix, we should continue as follows:

- All entries above the lowest (rightmost) leading 1 must be zero.
- All entries above the next leading 1 must be zero.
- $\cdots$

We can illustrate this procedure on a sample 4×4 matrix as follows:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & \bullet \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix we obtained at the end of this procedure is in REF. Now in the second part, we reduce the matrix further into RREF:

$$\begin{bmatrix} 1 & \bullet & \bullet & 0 \\ 0 & 1 & \bullet & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \bullet & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EXERCISES

Are the following matrices in row-echelon form (REF)?

27–1) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–2) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–3) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

27–4) $\begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

27–5) $\begin{bmatrix} 1 & -5 & 3 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$

27–6) $\begin{bmatrix} 1 & 4 & -8 & 40 \\ 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–7) $\begin{bmatrix} 1 & 4 & -8 & 2 & 0 \\ 0 & 1 & 5 & -7 & 6 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Are the following matrices in reduced row-echelon form (RREF)?

27–8) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–9) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

27–10) $\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

27–11) $\begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$

27–12) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–13) $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–14) $\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 1 & 0 & 8 & 3 \end{bmatrix}$

Using elementary row operations, reduce the following matrices to REF.

27–15) $\begin{bmatrix} 1 & 4 \\ 2 & 12 \end{bmatrix}$

27–16) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 4 \\ 2 & 7 & 13 \end{bmatrix}$

27–17) $\begin{bmatrix} 3 & 0 & 6 & -6 \\ 0 & 1 & 0 & -3 \\ 2 & -3 & 4 & 5 \end{bmatrix}$

27–18) $\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 2 & 0 & 1 \\ 1 & 4 & 1 & 4 \\ -4 & 1 & 3 & 4 \end{bmatrix}$

27–19) $\begin{bmatrix} 3 & 21 \\ 2 & 18 \end{bmatrix}$

27–20) $\begin{bmatrix} 4 & -8 & 16 \\ -8 & 15 & -44 \\ -3 & 7 & -4 \end{bmatrix}$

27–21) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 6 & 3 & 3 & 9 \\ 8 & 5 & 9 & 21 \end{bmatrix}$

27–22) $\begin{bmatrix} 2 & 3 \\ 1 & 7 \\ 5 & -6 \end{bmatrix}$

Using elementary row operations, reduce the following matrices to RREF.

27–23) $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 5 \end{bmatrix}$

27–24) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 7 & 7 & 4 \end{bmatrix}$

27–25) $\begin{bmatrix} 2 & 10 & 5 & -1 \\ 0 & 0 & 1 & 4 \\ 1 & 5 & 3 & 2 \end{bmatrix}$

27–26) $\begin{bmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 3 & -4 \\ 1 & 2 & 5 & -6 \\ 2 & 1 & 1 & 0 \end{bmatrix}$

27–27) $\begin{bmatrix} 3 & -6 & 12 \\ -1 & 14 & 92 \\ 1 & 3 & 51 \end{bmatrix}$

27–28) $\begin{bmatrix} 1 & 2 & -2 & 6 & -28 \\ 4 & 7 & -7 & 22 & -98 \\ 2 & 9 & -6 & 25 & -111 \end{bmatrix}$

27–29) $\begin{bmatrix} 1 & 2 & 16 \\ 4 & 7 & 50 \end{bmatrix}$

27–30) $\begin{bmatrix} 5 & 20 & 75 \\ 2 & 16 & 46 \\ 3 & 11 & 43 \end{bmatrix}$

ANSWERS

27–1) Yes

27–2) Yes

27–3) No

27–4) No

27–5) Yes

27–6) No

27–7) No

27–8) Yes

27–9) Yes

27–10) Yes

27–11) No

27–12) Yes

27–13) Yes

27–14) No

You may find slightly different results for the exercises on this page. REF form of a matrix depends on the operations we choose, but RREF form is unique.

27–15) $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

27–16) $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

27–17) $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–18) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–19) $\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$

27–20) $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$

27–21) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

27–22) $\begin{bmatrix} 1 & 7 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

27–23) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–24) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–25) $\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27–26) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

27–27) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27–28) $\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & -9 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix}$

27–29) $\begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & 14 \end{bmatrix}$

27–30) $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Chapter 28

Systems of Linear Equations

Equations and Solutions: A system of n linear equations in m unknowns is:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m & = & b_2 \\ \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m & = & b_n \end{array}$$

For example,

$$\begin{array}{rcl} 2x_1 - 3x_2 & = & 7 \\ x_1 + x_2 & = & 6 \end{array}$$

is a system of 2 linear equations in 2 unknowns. We can easily find the solution as: $x_1 = 5, \quad x_2 = 1.$

$$\begin{array}{rcl} 2x_1 + 2x_2 & = & 10 \\ x_1 + x_2 & = & 6 \end{array}$$

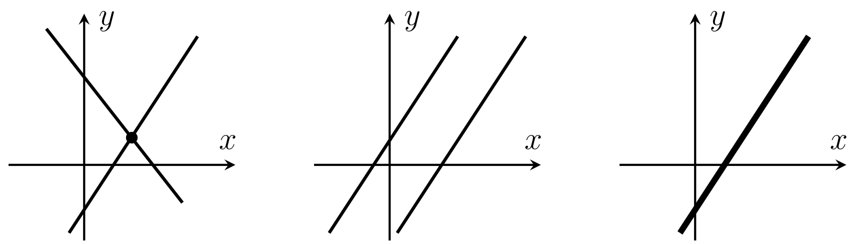
is another system. It has no solution, in other words, it is **inconsistent**. (Can you see why?)

Two important questions about systems of equations are:

- Is there a solution? (Is the system consistent?)
- Supposing there is a solution, are there other solutions or is the solution unique?

A geometric interpretation will help us analyze this problem. The solution of 2 linear equations in 2 unknowns can be considered as the point of intersection of 2 lines in plane. Clearly, there are three possibilities:

- The lines intersect at a single point. (Unique solution)
- The lines are paralel. (No solution)
- The lines are identical. (Infinitely many solutions)



Gaussian Elimination:

The method of Gaussian elimination can be summarized as:

- Represent the system of equations by an augmented matrix.
- Using row operations, obtain row echelon form (REF) of this matrix.
- Use back-substitution to find the unknowns.

The main idea is that, each row represents one equation. So, row operations simplify the system but do not change the solution.

For example, consider:

$$\begin{aligned} x - 3y &= -1 \\ 2x + y &= 12 \end{aligned}$$

The augmented matrix representing this system is:

$$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 2 & 1 & 12 \end{array}\right]$$

Reduction to REF gives:

$$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

The system of equations is:

$$\begin{aligned} x - 3y &= -1 \\ y &= 2 \end{aligned}$$

Back substitution gives:

$$y = 2 \quad \Rightarrow \quad x = 5.$$

The system

$$\begin{aligned} x + 5y &= 6 \\ 2x + 10y &= 15 \end{aligned}$$

has no solution. We can see this using Gaussian elimination as follows:

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 2 & 10 & 15 \end{array}\right]$$

Subtract 2 times the first row from the second row:

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 0 & 0 & 3 \end{array}\right]$$

Back substitution gives:

$$\begin{aligned} 0 &= 3 \\ x + 5y &= 6 \end{aligned}$$

But obviously $0 \neq 3$. We have a contradiction. Therefore there is no solution. The similar system

$$\begin{aligned} x + 5y &= 6 \\ 2x + 10y &= 12 \end{aligned}$$

has infinitely many solutions. Using Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 2 & 10 & 12 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

Back substitution gives:

$$\begin{aligned} 0 &= 0 \\ x + 5y &= 6 \end{aligned}$$

We can choose y in any way we like. It is a free parameter. So the solution is:

$$x = 6 - 5y$$

Example 28–1: Solve the system of equations

$$\begin{aligned} 2x+8y+4z &= 14 \\ 2x+7y+3z &= 7 \\ -5x-18y-5z &= 15 \end{aligned}$$

using Gaussian elimination.

Solution: First, we will use an augmented matrix to express the system of equations in a simple way:

$$\left[\begin{array}{rrr|r} 2 & 8 & 4 & 14 \\ 2 & 7 & 3 & 7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

Then we will use row operations to reduce this matrix to REF:

$$R_1 \to \frac{1}{2}R_1 \qquad \left[\begin{array}{rrr|r} 1 & 4 & 2 & 7 \\ 2 & 7 & 3 & 7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1 \qquad \left[\begin{array}{rrr|r} 1 & 4 & 2 & 7 \\ 0 & -1 & -1 & -7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

$$R_2 \to -R_2 \qquad \left[\begin{array}{rrr|r} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ -5 & -18 & -5 & 15 \end{array}\right]$$

$$R_3 \to R_3 + 5R_1 \qquad \left[\begin{array}{rrr|r} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ 0 & 2 & 5 & 50 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \qquad \left[\begin{array}{rrr|r} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 3 & 36 \end{array}\right]$$

$$R_3 \to \frac{1}{3}R_3 \qquad \left[\begin{array}{rrr|r} 1 & 4 & 2 & 7 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

This matrix is in REF. The system of equations represented by that is:

$$\begin{aligned} x+4y+2z &= 7 \\ y+z &= 7 \\ z &= 12 \end{aligned}$$

At this point, we start the back substitution:

$$z = 12$$

$$y+12=7 \Rightarrow y=-5$$

$$x-20+24=7 \Rightarrow x=3$$

Clearly, the solution is unique.

Example 28–2: Solve the following system of equations:

$$\begin{aligned} x+z &= 4 \\ 5x+4y-7z &= 16 \\ 2x-3y+11z &= 11 \end{aligned}$$

Solution: The augmented matrix is:

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 5 & 4 & -7 & 16 \\ 2 & -3 & 11 & 11 \end{array}\right]$$

Using row operations, we obtain:

$$\begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 4 & -12 & -4 \\ 0 & -3 & 9 & 3 \end{array}\right]$$

$$R_2 \to \frac{1}{4}R_2 \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & 9 & 3 \end{array}\right]$$

$$R_3 \to R_3 + 3R_2 \quad \left[\begin{array}{rrr|r} 1 & 0 & 1 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The back substitution gives:

$$\begin{aligned} y-3z &= -1 \quad \Rightarrow \quad y = -1+3z \\ x+z &= 4 \quad \Rightarrow \quad x = 4-z \end{aligned}$$

Here, z is a free parameter. There are infinitely many solutions.

Example 28–3: Solve the following system of equations:

$$\begin{aligned} x+4y+z &= -1 \\ 2x+10y &= 8 \\ x+3y+2z &= -5 \end{aligned}$$

Solution: The augmented matrix is:

$$\left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 2 & 10 & 0 & 8 \\ 1 & 3 & 2 & -5 \end{array}\right]$$

Using row operations, we obtain:

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 0 & 2 & -2 & 10 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$R_2 \to \frac{1}{2}R_2 \quad \left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 0 & 1 & -1 & 5 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \quad \left[\begin{array}{rrr|r} 1 & 4 & 1 & -1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The back substitution gives:

$$\begin{aligned} 0 &= 1 \\ y-z &= 5 \\ x+4y+z &= -1 \end{aligned}$$

$0=1$ is impossible, a contradiction. Therefore there are no solutions. The system is inconsistent.

Gauss-Jordan Elimination:
Gauss–Jordan elimination is similar to Gaussian elimination, but instead of stopping at REF, we go all the way to RREF. So, the reduction of the augmented matrix takes longer. But as an advantage, we do not have to use back-substitution. The rightmost column gives the solution directly.

Example 28–4: Solve the following system of equations:

$$\begin{aligned} y + 11z &= -3 \\ x + 3y + 10z &= 19 \\ x + y + 3z &= 10 \end{aligned}$$

Solution: The augmented matrix is:

$$\left[\begin{array}{ccc|c} 0 & 1 & 11 & -3 \\ 1 & 3 & 10 & 19 \\ 1 & 1 & 3 & 10 \end{array}\right]$$

Using row operations, we obtain:

$$R_1 \longleftrightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 1 & 3 & 10 & 19 \\ 0 & 1 & 11 & -3 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 2 & 7 & 9 \\ 0 & 1 & 11 & -3 \end{array}\right]$$

$$R_2 \longleftrightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 11 & -3 \\ 0 & 2 & 7 & 9 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 11 & -3 \\ 0 & 0 & -15 & 15 \end{array}\right]$$

$$R_3 \to -\frac{1}{15} R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 11 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This is in REF but we continue:

$$R_2 \to R_2 - 11R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 10 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - 3R_3 \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This augmented matrix is in RREF. The solution is:

$$\begin{aligned} x &= 5 \\ y &= 8 \\ z &= -1 \end{aligned}$$

Clearly, the system has unique solution.

Example 28–5: Solve the system of equations:

$$\begin{aligned} x_1 + x_2 - 2x_4 - 2x_5 &= 5 \\ -x_1 + 2x_2 + x_3 - x_4 - 5x_5 &= 7 \\ 3x_1 - 4x_3 + 6x_4 + x_5 &= 3 \end{aligned}$$

using Gauss-Jordan elimination.

Solution: The augmented matrix is:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ -1 & 2 & 1 & -1 & -5 & 7 \\ 3 & 0 & -4 & 6 & 1 & 3 \end{array}\right]$$

Using row operations, we obtain:

$$R_2 \to R_2 + R_1 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 3 & 1 & -3 & -7 & 12 \\ 3 & 0 & -4 & 6 & 1 & 3 \end{array}\right]$$

$$R_3 \to R_3 - 3R_1 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 3 & 1 & -3 & -7 & 12 \\ 0 & -3 & -4 & 12 & 7 & -12 \end{array}\right]$$

$$R_2 \to \frac{1}{3}R_2 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & \frac{1}{3} & -1 & -\frac{7}{3} & 4 \\ 0 & -3 & -4 & 12 & 7 & -12 \end{array}\right]$$

$$R_3 \to R_3 + 3R_2 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & \frac{1}{3} & -1 & -\frac{7}{3} & 4 \\ 0 & 0 & -3 & 9 & 0 & 0 \end{array}\right]$$

$$R_3 \to -\frac{1}{3}R_3 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & \frac{1}{3} & -1 & -\frac{7}{3} & 4 \\ 0 & 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

This matrix is in REF but we continue the reduction:

$$R_2 \to R_2 - \tfrac{1}{3}R_3 \qquad \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & -2 & -2 & 5 \\ 0 & 1 & 0 & 0 & -\frac{7}{3} & 4 \\ 0 & 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \qquad \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -2 & \frac{1}{3} & 1 \\ 0 & 1 & 0 & 0 & -\frac{7}{3} & 4 \\ 0 & 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

This augmented matrix is in RREF.

$$\begin{aligned} x_1 - 2x_4 + \tfrac{1}{3}x_5 &= 1 \\ x_2 - \tfrac{7}{3}x_5 &= 4 \\ x_3 - 3x_4 &= 0 \end{aligned}$$

There are infinitely many solutions. We can choose x_4 and x_5 in an arbitrary way. The solution can be expressed as:

$$\begin{aligned} x_1 &= 1 + 2s - t \\ x_2 &= 4 + 7t \\ x_3 &= 3s \\ x_4 &= s \\ x_5 &= 3t \end{aligned}$$

where s and t are free parameters.

EXERCISES

Solve the following systems of equations:

28–1)
$$\begin{aligned} 2x+3y &= -6 \\ x-2y &= 11 \end{aligned}$$

28–2)
$$\begin{aligned} 2x+6y &= 24 \\ -x-3y &= -6 \end{aligned}$$

28–3)
$$\begin{aligned} 4x+5y &= -2 \\ -8x-10y &= 4 \end{aligned}$$

28–4)
$$\begin{aligned} 3x+5y &= 0 \\ x+5y &= -10 \end{aligned}$$

28–5)
$$\begin{aligned} 5x+2y &= 4 \\ 3x-y &= 20 \end{aligned}$$

28–6)
$$\begin{aligned} -x+3y &= 0 \\ 20x+10y &= 7 \end{aligned}$$

28–7)
$$\begin{aligned} x-y &= -4 \\ 10x-4y &= 2 \end{aligned}$$

28–8)
$$\begin{aligned} x+3y &= -15 \\ 3x-y &= 15 \end{aligned}$$

28–9)
$$\begin{aligned} 4x-8y &= 10 \\ 12x-24y &= 20 \end{aligned}$$

28–10)
$$\begin{aligned} x+5y &= 8 \\ 4x+20y &= 32 \end{aligned}$$

28–11)
$$\begin{aligned} 2x_1+6x_2 &= 13 \\ -x_1+4x_2 &= 11 \end{aligned}$$

28–12)
$$\begin{aligned} x_1-20x_2 &= 8 \\ 5x_1-x_2 &= 7 \end{aligned}$$

Solve the following systems of equations:

28–13)
$$\begin{aligned} x_1+2x_2-x_3 &= 6 \\ 2x_1+x_2+4x_3 &= 9 \\ -x_1-3x_2+5x_3 &= -5 \end{aligned}$$

28–14)
$$\begin{aligned} x_1+2x_2+2x_3 &= 11 \\ x_1+3x_2+13x_3 &= 10 \\ -x_1+2x_2+12x_3 &= 0 \end{aligned}$$

28–15)
$$\begin{aligned} x_1+4x_2+2x_3 &= -3 \\ x_1+x_2-x_3 &= 3 \\ -2x_1-4x_3 &= 6 \end{aligned}$$

28–16)
$$\begin{aligned} x_1+2x_2-x_3+x_4 &= 3 \\ 2x_1+x_2+x_3+x_4 &= 4 \\ x_1-x_2+2x_3 &= 1 \end{aligned}$$

28–17)
$$\begin{aligned} x_1+4x_3 &= 1 \\ 2x_1+x_2+3x_3 &= 5 \\ 3x_1+2x_2+2x_3 &= 9 \end{aligned}$$

28–18)
$$\begin{aligned} x_1+2x_2-5x_3 &= -1 \\ x_1+3x_2-7x_3 &= 0 \\ -x_1+x_2-2x_3 &= 3 \end{aligned}$$

28–19)
$$\begin{aligned} x_1-x_3 &= 3 \\ -x_1+2x_2-x_3+2x_4 &= -6 \\ 2x_1+3x_2+x_3 &= 9 \\ 4x_1+4x_3+10x_4 &= 15 \end{aligned}$$

28–20)
$$\begin{aligned} x_1+x_2+3x_3-x_4 &= 3 \\ -x_1+2x_2-3x_3+x_4 &= 0 \\ 5x_1+4x_2+10x_4 &= -1 \\ 7x_1+4x_2+6x_3+8x_4 &= 0 \end{aligned}$$

Solve the following homogeneous systems of equations:

28–21) $$\begin{aligned} 3x_1 + 2x_2 &= 0 \\ x_1 - 4x_2 &= 0 \end{aligned}$$

28–22) $$\begin{aligned} 2x_1 - 5x_2 &= 0 \\ -6x_1 + 15x_2 &= 0 \end{aligned}$$

28–23) $$\begin{aligned} x_1 - 3x_2 &= 0 \\ 4x_1 + 7x_2 &= 0 \\ 2x_1 + 8x_2 &= 0 \end{aligned}$$

28–24) $$\begin{aligned} x_1 - 4x_2 - 7x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

28–25) $$\begin{aligned} x_1 - x_2 - x_3 &= 0 \\ 4x_1 + 2x_2 - 13x_3 &= 0 \\ 2x_1 + 4x_2 - 11x_3 &= 0 \end{aligned}$$

28–26) $$\begin{aligned} -x_1 + 2x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 - 6x_3 &= 0 \\ 2x_1 + x_2 + 7x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

28–27) $$\begin{aligned} 2x_1 + 3x_2 - 7x_3 - 7x_4 &= 0 \\ 3x_1 - 6x_2 + 21x_4 &= 0 \\ -x_1 - 5x_2 + 7x_3 + 14x_4 &= 0 \end{aligned}$$

28–28) $$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 - 3x_2 - 9x_4 &= 0 \\ x_1 + 4x_2 + x_3 - x_4 &= 0 \end{aligned}$$

Solve the following systems of equations:

28–29) $$\begin{aligned} x + 2y + 3z &= 9 \\ 2y + z &= 4 \\ x + 2y + 4z &= 11 \end{aligned}$$

28–30) $$\begin{aligned} x + y - z &= 0 \\ 2x + 5y + z &= 9 \\ x + 8y + 4z &= 13 \end{aligned}$$

28–31) $$\begin{aligned} 2x - y + 5z &= -2 \\ 2y + 3z &= 16 \\ x + y - z &= 11 \end{aligned}$$

28–32) $$\begin{aligned} 3x + y - z &= 5 \\ x - 2y + 8z &= -3 \\ 10x + y + 5z &= 10 \end{aligned}$$

28–33) $$\begin{aligned} x + 2y &= 4 \\ x + 3y + z &= 3 \\ 4x + 7y - z &= 17 \end{aligned}$$

28–34) $$\begin{aligned} x - 5y + z &= 4 \\ 3x - 12y + 4z &= 9 \\ 3y + z &= 0 \end{aligned}$$

28–35) $$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 4 \\ 3x_1 - 2x_2 + 3x_3 - 2x_4 &= 15 \\ 2x_1 - 2x_2 + 3x_3 - x_4 &= 10 \\ 3x_1 - 3x_2 + 4x_3 - x_4 &= 16 \end{aligned}$$

ANSWERS

28–1) $x = 3, \quad y = -4.$

28–2) There's no solution.

28–3) There are infinitely many solutions. y is arbitrary parameter and $x = -\frac{1}{2} - \frac{5}{4}y.$

28–4) $x = 5, \quad y = -3.$

28–5) $x = 4, \quad y = -8.$

28–6) $x = 0.3, \quad y = 0.1.$

28–7) $x = 3, \quad y = 7.$

28–8) $x = 3, \quad y = -6.$

28–9) There's no solution.

28–10) There are infinitely many solutions. y is arbitrary parameter and $x = 8 - 5y.$

28–11) $x_1 = -1, \quad x_2 = 2.5.$

28–12) $x_1 = \frac{4}{3}, \quad x_2 = -\frac{1}{3}.$

28–13) $x_1 = 1, \quad x_2 = 3, \quad x_3 = 1.$

28–14) $x_1 = 3, \quad x_2 = 4.5, \quad x_3 = -0.5.$

28–15) $x_1 = 1, \quad x_2 = 0, \quad x_3 = -2.$

28–16) There are infinitely many solutions given by:
$x_1 = \frac{5}{3} - s - r, \quad x_2 = \frac{2}{3} - s + r, \quad x_3 = r, \quad x_4 = 3s.$

28–17) There are infinitely many solutions. z is arbitrary parameter and $x = 1 - 4r, \quad y = 3 + 5r.$

28–18) $x_1 = -2, \quad x_2 = 3, \quad x_3 = 1.$

28–19) $x_1 = 4, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = -0.5.$

28–20) There's no solution.

28–21) Trivial solution.

28–22) $x_1 = 5r, \quad x_2 = 2r.$

28–23) Trivial solution.

28–24) $x_1 = 3r, \quad x_2 = -r, \quad x_3 = r.$

28–25) $x_1 = 5r, \quad x_2 = 3r, \quad x_3 = 2r.$

28–26) Trivial solution.

28–27) $x_1 = 2r - s, \quad x_2 = r + 3s, \quad x_3 = r, \quad x_4 = s.$

28–28) $x_1 = 3r, \quad x_2 = -r, \quad x_3 = 2r, \quad x_4 = r.$

28–29) $x = 1, \quad y = 1, \quad z = 2.$ Unique Solution.

28–30) $x = 5, \quad y = -1, \quad z = 4.$ Unique Solution.

28–31) $x = 3, \quad y = 8, \quad z = 0.$ Unique Solution.

28–32) No Solution.

28–33) $x = 6 + 2z, \quad y = -z - 1, \quad z$ is a free parameter.
Infinitely Many Solutions.

28–34) No Solution.

28–35) $x_1 = 7, \quad x_2 = 1, \quad x_3 = 0, \quad x_4 = 2.$
Unique Solution.

Chapter 29

Inverse Matrices

Let A be an $n \times n$ matrix. If there exists another $n \times n$ matrix B such that

$$AB = BA = I$$

then B is called the **inverse** of A. (Similarly, A is the inverse of B.) For example,

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -\frac{1}{2} \\ -3 & 1 \end{bmatrix}$$

are inverses of each other.

If a square matrix A has an inverse, we say A is **invertible**.

If A and B are invertible and same size, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^n)^{-1} = (A^{-1})^n = A^{-n}$
- $(A^T)^{-1} = (A^{-1})^T$

Example 29–1: Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if it exists.

Solution: Assume the inverse exists and it is $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$.

Using definition, we obtain:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} aw + by &= 1 \\ cw + dy &= 0 \end{aligned}$$

$$\begin{aligned} ax + bz &= 0 \\ cx + dz &= 1 \end{aligned}$$

We can solve for w, x, y, z using elimination.

The solution is: $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Therefore, the inverse of A exists if $ad - bc \neq 0$.

Matrix Inverse by Row Reduction:
Let A be a square matrix. To find A^{-1}, write the identity matrix I next to A:

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

Then use row operations to obtain:

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

(This is equivalent to finding RREF.)

If there is a row of zeros on the bottom (for the left part), that means A can not be reduced to I. In other words A is not invertible.

Example 29–2: Find the inverse of

$$A = \begin{bmatrix} 1 & 4 & 21 \\ 0 & -1 & -5 \\ 1 & 2 & 10 \end{bmatrix}$$

if it exists.

Solution: Let's start with:

$$\left[\begin{array}{rrr|rrr} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & -1 & -5 & 0 & 1 & 0 \\ 1 & 2 & 10 & 0 & 0 & 1 \end{array}\right]$$

and use row reduction:

$$R_3 \to R_3 - R_1 \qquad \left[\begin{array}{rrr|rrr} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & -1 & -5 & 0 & 1 & 0 \\ 0 & -2 & -11 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \to -R_2 \qquad \left[\begin{array}{rrr|rrr} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & -1 & 0 \\ 0 & -2 & -11 & -1 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 + 2R_2 \qquad \left[\begin{array}{rrr|rrr} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 1 \end{array}\right]$$

$$R_3 \to -R_3 \qquad \left[\begin{array}{rrr|rrr} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

$$R_2 \to R_2 - 5R_3 \qquad \left[\begin{array}{rrr|rrr} 1 & 4 & 21 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & -11 & 5 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

$$R_1 \to R_1 - 21R_3 \qquad \left[\begin{array}{rrr|rrr} 1 & 4 & 0 & -20 & -42 & 21 \\ 0 & 1 & 0 & -5 & -11 & 5 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

$$R_1 \to R_1 - 4R_2 \qquad \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -5 & -11 & 5 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ -5 & -11 & 5 \\ 1 & 2 & -1 \end{bmatrix}$.

Example 29–3: Find the inverse of $B = \begin{bmatrix} 1 & 2 & -2 & -1 \\ 1 & 3 & 5 & 7 \\ 0 & 0 & 2 & 2 \\ 2 & 4 & 1 & 4 \end{bmatrix}$.

Solution: Start with: $\left[\begin{array}{cccc|cccc} 1 & 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 5 & 7 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_4 \to R_4 - 2R_1 \end{array} \quad \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_3 \to \frac{1}{2} R_3 \\ R_4 \to R_4 - 5R_3 \end{array} \quad \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 8 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & -\frac{5}{2} & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + R_4 \\ R_2 \to R_2 - 8R_4 \\ R_3 \to R_3 - R_4 \end{array} \quad \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & 0 & -1 & 0 & -\frac{5}{2} & 1 \\ 0 & 1 & 7 & 0 & 15 & 1 & 20 & -8 \\ 0 & 0 & 1 & 0 & 2 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 & -\frac{5}{2} & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 - 7R_3 \\ R_1 \to R_1 - 2R_2 \end{array} \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & \frac{11}{2} & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 & -\frac{5}{2} & 1 \end{array}\right]$$

$\Rightarrow$ B^{-1} is the right part of the matrix.

Example 29–4: Find the inverse of

$$C = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 5 & -6 \\ 7 & 8 & -12 \end{bmatrix}$$

if it exists.

Solution: Let's start with:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 3 & 5 & -6 & 0 & 1 & 0 \\ 7 & 8 & -12 & 0 & 0 & 1 \end{array}\right]$$

and use row reduction:

$$R_2 \to R_2 - 3R_1 \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 11 & -6 & -3 & 1 & 0 \\ 7 & 8 & -12 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 7R_1 \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 11 & -6 & -3 & 1 & 0 \\ 0 & 22 & -12 & -7 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 11 & -6 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

We have a row of zeros.

$\Rightarrow$ It is impossible to row reduce C to I.

$\Rightarrow$ C is not invertible.

Solution of Systems of Equations by Matrix Inverses:
Consider the system of n linear equations in m unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ \vdots \qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

We can represent this system as a matrix equation:

$$A\overrightarrow{x} = \overrightarrow{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad \overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad \overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

For the special case $n = m$, the matrix A is square. In this case, there is a unique solution if and only if A is invertible, where

$$\overrightarrow{x} = A^{-1}\overrightarrow{b}$$

If A is not invertible, there may be no solution or there may be infinitely many solutions, depending on $\overrightarrow{b}$.
The homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$ always has the trivial (zero) solution.

$$\overrightarrow{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If A is invertible, there is no other, nontrivial solution.

Example 29–5: Solve the following systems of equations:

$$\begin{aligned} x_1 + 4x_3 &= 6 \\ -2x_2 - x_3 &= 5 \\ 3x_1 + 5x_2 + 2x_3 &= -7 \end{aligned} \qquad \begin{aligned} y_1 + 4y_3 &= 50 \\ -2y_2 - y_3 &= 75 \\ 3y_1 + 5y_2 + 2y_3 &= 100 \end{aligned}$$

Solution: We can solve multiple systems having the same coefficient matrix by using a single augmented matrix. The given systems can be written as:

$A\overrightarrow{x} = \overrightarrow{b}$ and $A\overrightarrow{y} = \overrightarrow{c}$, where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 3 & 5 & 2 \end{bmatrix}, \quad \overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$\overrightarrow{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \overrightarrow{b} = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}, \quad \overrightarrow{c} = \begin{bmatrix} 50 \\ 75 \\ 100 \end{bmatrix}.$$

Using row reduction on $\left[\begin{array}{c|c} A & I \end{array}\right]$ we find:

$$A^{-1} = \frac{1}{25}\begin{bmatrix} 1 & 20 & 8 \\ -3 & -10 & 1 \\ 6 & -5 & -2 \end{bmatrix}.$$

Now we can solve both systems easily as:

$$\overrightarrow{x} = A^{-1}\overrightarrow{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \text{ and}$$

$$\overrightarrow{y} = A^{-1}\overrightarrow{c} = \begin{bmatrix} 94 \\ -32 \\ -11 \end{bmatrix}$$

EXERCISES

Find the inverse of each matrix, if it exists:

29–1) $\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix}$

29–2) $\begin{bmatrix} 13 & 5 \\ 5 & 1 \end{bmatrix}$

29–3) $\begin{bmatrix} 4 & 3 \\ 12 & 9 \end{bmatrix}$

29–4) $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

29–5) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 5 & 2 \end{bmatrix}$

29–6) $\begin{bmatrix} 4 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

29–7) $\begin{bmatrix} 7 & 2 & 3 \\ -4 & 1 & -5 \\ -1 & 4 & -7 \end{bmatrix}$

29–8) $\begin{bmatrix} 11 & 6 & -5 \\ -8 & 2 & 12 \\ 1 & -9 & -13 \end{bmatrix}$

Find the inverse of each matrix, if it exists:

29–9) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

29–10) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

29–11) $\begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}$

29–12) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 11 & 18 \\ 2 & 6 & 11 \end{bmatrix}$

29–13) $\begin{bmatrix} 2 & 4 & 1 \\ 6 & 6 & -20 \\ 1 & 5 & 12 \end{bmatrix}$

29–14) $\begin{bmatrix} 1 & 4 & 6 \\ -1 & 2 & 8 \\ 5 & 3 & 1 \end{bmatrix}$

29–15) $\begin{bmatrix} 5 & 2 & 0 \\ 1 & -2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$

29–16) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 0 & -5 \end{bmatrix}$

Find the inverse of each matrix, if it exists:

29–17) $\begin{bmatrix} 4 & 0 & 0 \\ -2 & 3 & 0 \\ 1 & -9 & 1/2 \end{bmatrix}$

29–18) $\begin{bmatrix} 0.5 & -1 & 5 \\ 1 & 2 & 4 \\ 1 & 6 & 0 \end{bmatrix}$

29–19) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 1 & 3 & -2 \\ 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 6 \end{bmatrix}$

29–20) $\begin{bmatrix} 2 & -2 & 4 & 1 \\ 2 & -1 & 0 & 4 \\ 2 & -3 & 6 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix}$

29–21) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 1 & 3 & -1 \\ 1 & 1 & 2 & 1 \\ -2 & 0 & 1 & 5 \end{bmatrix}$

29–22) $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 0 & 1 & 2 & 4 \\ 2 & 1 & 0 & 4 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

Solve the following systems of equations using inverse of the coefficient matrix:

29–23) $$\begin{aligned} 3x_1 + 4x_2 - 2x_3 &= 11 \\ -x_1 + 2x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

29–24) $$\begin{aligned} 3x_1 + 4x_2 - 2x_3 &= 16 \\ -x_1 + 2x_2 + 2x_3 &= 18 \\ x_1 + 2x_2 - x_3 &= 7 \end{aligned}$$

29–25) $$\begin{aligned} 5x_1 + 8x_2 + 5x_3 &= 9 \\ x_1 + 2x_2 + x_3 &= 3 \\ x_2 + x_3 &= 4 \end{aligned}$$

29–26) $$\begin{aligned} 5x_1 + 8x_2 + 5x_3 &= 2 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + x_3 &= 1 \end{aligned}$$

29–27) $$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 5x_2 + x_3 &= 15 \\ -x_1 + x_3 &= -2 \end{aligned}$$

29–28) $$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -x_1 - x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= -2 \end{aligned}$$

29–29) $$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + x_2 + 3x_3 - 2x_4 &= -1 \\ x_1 - x_2 + 2x_3 - 3x_4 &= -11 \\ x_2 - 2x_3 + 6x_4 &= 25 \end{aligned}$$

29–30) $$\begin{aligned} 2x_1 - 2x_2 + 4x_3 + x_4 &= 5 \\ 2x_1 - x_2 + 4x_4 &= 9 \\ 2x_1 - 3x_2 + 5x_3 + x_4 &= 4 \\ x_2 - 2x_4 &= -1 \end{aligned}$$

ANSWERS

29–1) $\frac{1}{8}\begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix}$

29–2) $\frac{1}{12}\begin{bmatrix} -1 & 5 \\ 5 & -13 \end{bmatrix}$

29–3) Inverse does not exist.

29–4) $\frac{1}{3}\begin{bmatrix} -5 & 2 & -2 \\ 4 & -1 & 1 \\ -5 & 2 & 1 \end{bmatrix}$

29–5) $\begin{bmatrix} 2.5 & 0.5 & -1 \\ 0.5 & 0.5 & 0 \\ -2.5 & -1.5 & 1 \end{bmatrix}$

29–6) $\frac{1}{10}\begin{bmatrix} -2 & 0 & 6 \\ -2 & 5 & 1 \\ 6 & 0 & -8 \end{bmatrix}$

29–7) Inverse does not exist.

29–8) Inverse does not exist.

29–9) $\frac{1}{2}\begin{bmatrix} 6 & -2 & -4 \\ 1 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}$

29–10) $\frac{1}{2}\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & -1 \\ -1 & 2 & -1 \end{bmatrix}$

29–11) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$

29–12) $\frac{1}{6}\begin{bmatrix} 13 & -9 & 10 \\ 3 & 3 & -6 \\ -4 & 0 & 2 \end{bmatrix}$

29–13) Inverse does not exist.

29–14) $\frac{1}{64}\begin{bmatrix} -22 & 14 & 20 \\ 41 & -29 & -14 \\ -13 & 17 & 6 \end{bmatrix}$

29–15) $\frac{1}{5}\begin{bmatrix} -1 & 2 & 2 \\ 5 & -5 & -5 \\ 11 & -7 & -12 \end{bmatrix}$

29–16) Inverse does not exist.

29–17) $\frac{1}{12}\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 30 & 72 & 24 \end{bmatrix}$

29–18) $\frac{1}{4}\begin{bmatrix} -24 & 30 & -14 \\ 4 & -5 & 3 \\ 4 & -4 & 2 \end{bmatrix}$

29–19) $\frac{1}{7}\begin{bmatrix} 15 & -12 & 28 & 10 \\ -4 & 6 & -14 & -5 \\ -17 & 15 & -28 & -9 \\ -5 & 4 & -7 & -1 \end{bmatrix}$

29–20) $\frac{1}{2}\begin{bmatrix} 6 & -1 & -4 & -1 \\ -12 & 4 & 8 & 6 \\ -7 & 2 & 5 & 3 \\ -6 & 2 & 4 & 2 \end{bmatrix}$

29–21) Inverse does not exist.

29–22) $\frac{1}{18}\begin{bmatrix} -7 & -7 & 5 & 8 \\ 2 & 2 & -4 & 8 \\ -7 & 2 & -4 & 8 \\ 3 & 3 & 3 & -6 \end{bmatrix}$

29–23) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

29–24) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}$

29–25) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$

29–26) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

29–27) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

29–28) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

29–29) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$

29–30) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Chapter 30

Determinants

Determinants by Cofactors:
The determinant of a 1×1 matrix is itself. For any other square matrix, we define the determinant recursively.
We will use the notation $\left|A\right|$ to denote the determinant of the square matrix A.

Cofactor: Let A be an $n \times n$ matrix. Let's delete the i^{th} row and j^{th} column, and calculate the determinant of the remaining $(n-1) \times (n-1)$ submatrix. Call it M_{ij}. The cofactor of the entry a_{ij} is:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, for the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

cofactor of a is d and the cofactor of b is $-c$.
For the matrix

$$\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

cofactor of a is $\begin{vmatrix} q & r \\ y & z \end{vmatrix}$, cofactor of b is $-\begin{vmatrix} p & r \\ x & z \end{vmatrix}$ and cofactor of q is $\begin{vmatrix} a & c \\ x & z \end{vmatrix}$.

The determinant of any other square matrix is defined as:

$$\left|A\right| = \det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in}, \quad \text{or}$$
$$\left|A\right| = \det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$

In other words:

- Choose a row (or column).
- Find the cofactor of each entry on that row (or column).
- Multiply the entries by the cofactors and add.

This is called the cofactor expansion. For example, the determinant of a 2×2 matrix is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(Do you remember this expression from previous chapters?)

If we use the first row to find the determinant of a 3×3 matrix, we obtain:

$$\begin{aligned} \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} &= a\begin{vmatrix} q & r \\ y & z \end{vmatrix} - b\begin{vmatrix} p & r \\ x & z \end{vmatrix} + c\begin{vmatrix} p & q \\ x & y \end{vmatrix} \\ &= aqz - ayr - bpz + bxr + cpy - cqx \end{aligned}$$

If we choose the second column, we obtain:

$$\begin{aligned} \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} &= -b\begin{vmatrix} p & r \\ x & z \end{vmatrix} + q\begin{vmatrix} a & c \\ x & z \end{vmatrix} - y\begin{vmatrix} a & c \\ p & r \end{vmatrix} \\ &= -bpz + bxr + aqz - cqx - ayr + cpy \end{aligned}$$

which is identical. It does not matter which one we choose. Note that in each term, there's one and only one entry from each row and each column. Therefore the determinant of an $n \times n$ matrix requires $n!$ terms.

Example 30–1: Find the determinant of $A = \begin{bmatrix} 4 & 9 & 5 \\ 1 & 2 & 0 \\ -3 & 6 & 0 \end{bmatrix}$.

Solution: Using the third column, we obtain:

$$\begin{aligned} |A| &= 5\begin{vmatrix} 1 & 2 \\ -3 & 6 \end{vmatrix} . \\ &= 5\big(6 - (-6)\big) \\ &= 60 \end{aligned}$$

Example 30–2: Find determinant of $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 0 & 0 & 4 \\ 8 & -1 & 3 & 0 \\ -5 & 1 & 2 & -7 \end{bmatrix}$.

Solution: Using the second row, we obtain:

$$\begin{aligned} |A| &= -2\begin{vmatrix} 2 & -1 & 3 \\ -1 & 3 & 0 \\ 1 & 2 & -7 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 & -1 \\ 8 & -1 & 3 \\ -5 & 1 & 2 \end{vmatrix} . \\ &= (-2) \cdot (-50) + 4 \cdot (-70) \\ &= -180 \end{aligned}$$

Example 30–3: Find the determinant of $A = \begin{bmatrix} 5 & 48 & 7 & 12 \\ 0 & 3 & 10 & -3 \\ 0 & 0 & 2 & 99 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution: Using the first column, we obtain:

$$\begin{aligned} |A| &= 5\begin{vmatrix} 3 & 10 & -3 \\ 0 & 2 & 99 \\ 0 & 0 & 1 \end{vmatrix} . \\ &= 5 \cdot 3\begin{vmatrix} 2 & 99 \\ 0 & 1 \end{vmatrix} \\ &= 5 \cdot 3 \cdot 2 \cdot 1 \\ &= 30 \end{aligned}$$

Only the diagonal entries matter.

Inverses by Adjoint Matrices: Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}. If we replace each entry by its cofactor and then transpose the matrix, we obtain the adjoint. In other words, the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

is called the **adjoint** of A.
Do not forget that we have a sign of $(-1)^{i+j}$ in front of each entry. These signs are arranged as follows:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example, the adjoint of $A = \begin{bmatrix} 8 & 4 & 0 \\ 1 & 5 & 7 \\ 2 & 3 & 6 \end{bmatrix}$ is:

$$\begin{bmatrix} 9 & 8 & -7 \\ -24 & 48 & -16 \\ 28 & -56 & 36 \end{bmatrix}^T = \begin{bmatrix} 9 & -24 & 28 \\ 8 & 48 & -56 \\ -7 & -16 & 36 \end{bmatrix}$$

Theorem: If A is an invertible matrix, then $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

Example 30–4: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ using the above formula.

Solution:
$$\begin{aligned} A^{-1} &= \frac{\operatorname{adj}(A)}{\det(A)} \\ &= \frac{\begin{bmatrix} \begin{vmatrix} 5 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \end{bmatrix}^T}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 0 & 1 & 4 \end{vmatrix}} \\ &= \frac{\begin{bmatrix} 19 & -8 & 2 \\ -5 & 4 & -1 \\ -13 & 5 & 1 \end{bmatrix}^T}{9} \\ &= \frac{1}{9}\begin{bmatrix} 19 & -5 & -13 \\ -8 & 4 & 5 \\ 2 & -1 & 1 \end{bmatrix} \end{aligned}$$

Example 30–5: Find the inverse of the matrix $B = \begin{bmatrix} 7 & 5 & 8 \\ 2 & 1 & 0 \\ 6 & 4 & 3 \end{bmatrix}$

Solution: Let's find the cofactor of each entry:

Cofactor of 7: $C_{11} = \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix} = 3$

Cofactor of 5: $C_{12} = -\begin{vmatrix} 2 & 0 \\ 6 & 3 \end{vmatrix} = -6$

Cofactor of 8: $C_{13} = \begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} = 2$

Cofactor of 2: $C_{21} = -\begin{vmatrix} 5 & 8 \\ 4 & 3 \end{vmatrix} = 17$

Cofactor of 1: $C_{22} = \begin{vmatrix} 7 & 8 \\ 6 & 3 \end{vmatrix} = -27$

Cofactor of 0: $C_{23} = -\begin{vmatrix} 7 & 5 \\ 6 & 4 \end{vmatrix} = 2$

Cofactor of 6: $C_{31} = \begin{vmatrix} 5 & 8 \\ 1 & 0 \end{vmatrix} = -8$

Cofactor of 4: $C_{32} = -\begin{vmatrix} 7 & 8 \\ 2 & 0 \end{vmatrix} = 16$

Cofactor of 3: $C_{33} = \begin{vmatrix} 7 & 5 \\ 2 & 1 \end{vmatrix} = -3$

The determinant of B is:

$$\begin{aligned} |B| &= 7\begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 0 \\ 6 & 3 \end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 6 & 4 \end{vmatrix} \\ &= 21 - 30 + 16 = 7 \end{aligned}$$

The adjoint of B is:

$$\begin{aligned} \operatorname{adj} B &= \begin{bmatrix} 3 & -6 & 2 \\ 17 & -27 & 2 \\ -8 & 16 & -3 \end{bmatrix}^T \\ &= \begin{bmatrix} 3 & 17 & -8 \\ -6 & -27 & 16 \\ 2 & 2 & -3 \end{bmatrix} \end{aligned}$$

Therefore the inverse of B is:

$$B^{-1} = \frac{\operatorname{adj}(B)}{\det(B)} = \frac{1}{7}\begin{bmatrix} 3 & 17 & -8 \\ -6 & -27 & 16 \\ 2 & 2 & -3 \end{bmatrix}$$

Properties of the Determinant Function:
For $n \times n$ matrices:

- If A has a row (or column) of zeros, then $\det(A) = 0$.
- If A is an upper triangular, lower triangular or a diagonal matrix, then $\det(A)$ is the product of the entries on the main diagonal.
- The determinant of any identity matrix is 1.
- $\det(A) = \det(A^T)$
- A is invertible if and only if $\det(A) \neq 0$.
- $\det(AB) = \det(A)\det(B) \quad \Rightarrow \quad \det(A^{-1}) = \dfrac{1}{\det(A)}$.

EXERCISES

Find the determinants of the following matrices: (if defined)

30–1) $\begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

30–2) $\begin{bmatrix} 10 & 29 \\ 0 & 17 \end{bmatrix}$

30–3) $\begin{bmatrix} 5 & 9 & 7 \\ 12 & 0 & -3 \end{bmatrix}$

30–4) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 1 & 4 \\ 3 & 2 & -1 \end{bmatrix}$

30–5) $\begin{bmatrix} 8 & 2 & -1 \\ 1 & 0 & 0 \\ 9 & 6 & -5 \end{bmatrix}$

30–6) $\begin{bmatrix} -5 & 6 & 3 \\ 0 & 7 & 8 \\ 1 & 2 & 3 \end{bmatrix}$

30–7) $\begin{bmatrix} 3 & 1 & -2 & 0 \\ 5 & 1 & 3 & 0 \\ 4 & 2 & 1 & 6 \\ 8 & 2 & 3 & 0 \end{bmatrix}$

30–8) $\begin{bmatrix} 7 & 12 & 9 & 20 \\ 0 & 1 & 6 & 8 \\ 0 & 0 & 2 & 16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find the determinants of the following matrices: (if defined)

30–9) $\begin{bmatrix} 5 & x \\ 3 & 6 \end{bmatrix}$

30–10) $\begin{bmatrix} 4 & 13 & 2 \\ 0 & 1 & 0 \\ 8 & -7 & 9 \end{bmatrix}$

30–11) $\begin{bmatrix} a & b & c \\ 6 & 3 & -1 \\ 4 & 2 & 0 \end{bmatrix}$

30–12) $\begin{bmatrix} 7 & 5 & 3 \\ 14 & 10 & 6 \\ -4 & -2 & 1 \end{bmatrix}$

30–13) $\begin{bmatrix} 3 & 8 & 0 \\ x & y & z \\ 2 & 5 & 0 \end{bmatrix}$

30–14) $\begin{bmatrix} 2 & 4 & 0 & 3 \\ 0 & 7 & -8 & 12 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

30–15) $\begin{bmatrix} 7 & 11 \\ 56 & 93 \end{bmatrix}$

30–16) $\begin{bmatrix} 1 & 7 & 8 \\ 2 & 16 & 25 \\ 6 & 48 & 79 \end{bmatrix}$

Find the inverse of the following matrices using the adjoint formula:
(If possible)

30–17) $\begin{bmatrix} 10 & 8 \\ 7 & 6 \end{bmatrix}$

30–18) $\begin{bmatrix} 6 & 11 \\ 4 & 9 \end{bmatrix}$

30–19) $\begin{bmatrix} 10 & 7 \\ 30 & 21 \end{bmatrix}$

30–20) $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$

30–21) $\begin{bmatrix} 5 & 0 & 2 \\ 1 & -8 & 9 \\ 4 & -2 & 3 \end{bmatrix}$

30–22) $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 4 \\ 2 & 0 & 3 \end{bmatrix}$

30–23) $\begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & -4 \\ 0 & 2 & 0 \end{bmatrix}$

30–24) $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 5 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & 4 & -2 & 1 \end{bmatrix}$

ANSWERS

30–1) 32

30–2) 170

30–3) Determinant is defined only for square matrices.

30–4) -33

30–5) 4

30–6) 2

30–7) 24

30–8) 14

30–9) $30 - 3x$

30–10) 20

30–11) $2a - 4b$

30–12) 0

30–13) z

30–14) 28

30–15) 35

30–16) 8

30–17) $\frac{1}{4}\begin{bmatrix} 6 & -8 \\ -7 & 10 \end{bmatrix}$

30–18) $\frac{1}{10}\begin{bmatrix} 9 & -11 \\ -4 & 6 \end{bmatrix}$

30–19) Inverse does not exist.

30–20) $\frac{1}{24}\begin{bmatrix} -12 & 12 & 6 \\ 18 & -6 & -11 \\ 0 & 0 & 4 \end{bmatrix}$

30–21) $\frac{1}{30}\begin{bmatrix} -6 & -4 & 16 \\ 33 & 7 & -43 \\ 30 & 10 & -40 \end{bmatrix}$

30–22) $\frac{1}{15}\begin{bmatrix} -9 & 0 & 3 \\ -8 & 5 & -4 \\ 6 & 0 & 3 \end{bmatrix}$

30–23) Inverse does not exist.

30–24) $\frac{1}{27}\begin{bmatrix} 18 & 0 & -9 & 0 \\ -8 & 3 & 7 & 3 \\ -9 & 0 & 18 & 0 \\ 14 & -12 & 8 & 15 \end{bmatrix}$